

# Economic Production Lot Size Inventory Models under Learning and Forgetting Effects

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# Economic Production Lot Size Inventory Models under Learning and Forgetting Effects

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## Abstract

In this thesis, we tend to interpret the learning and forgetting phenomena in a closed view to give a clear idea of their importance in industrial engineering to determine the optimal production lot size. In doing so, we tend to review and comment on some of the recent studies concerning the effects of learning and forgetting on the optimal production quantity for single item. Then we shall study the effect of bounded learning on the production lot sizing problems for an infinite planning horizon where we present a general model for deteriorating items with time varying demand and deterioration rates, shortages are allowed and partially backordered. Both demand and deterioration rates are arbitrary functions of time. The production is subject to a full transmission of learning and the production rate is defined as the number of units produced per unit time. A closed form for the total associated cost as well as rigorous mathematical methods that lead to a minimum total cost of the underlying inventory system are derived. Besides, we shall study the effects of learning and forgetting on the production lot sizing problems for an infinite planning horizon. Items deteriorate while they are in storage, and both demand and deterioration rates are arbitrary functions of time. The system is subject to learning in the production stage and to forgetting while production ceased, so that the optimal manufactured quantity for any given cycle is assumed to be dependent on the amount of equivalent units of experience remembered at the beginning of the production run. We assume that the total forgetting is a function of the optimum production quantity. We

also provide a new definition for the production rate as the number of units remembered per unit time. A closed form for the total relevant costs is derived, and rigorous mathematical methods that guide to a minimum total cost of the underlying inventory system are introduced. An illustrative example, which explains the application of the theoretical results under partial transmission of learning and a numerical verification of this illustrative example, is also given. The numerical results clearly reflect the incorporated learning and forgetting effects in the proposed model.

## Introduction

The unused goods or materials available on hand at a particular time are known to be an Inventory. For instance, inventory may refer to tools, spare parts, equipments, amount of gases and oil still not refined, mines, water in dams, money in banks, and so forth. The term inventory as a problem is common to most organizations, since it is rare to find an organization that does not use, store, distribute, or sell materials of one kind or another. So, organizations are becoming increasingly aware of inventory, which is crucial to the success of many activities and is a part of their effective management. On the other hand, “learning and forgetting” as natural phenomena, are observable everywhere. Since they require dealing with any assignment involving some previous experience. For instance, if an individual is accustomed to do a certain assignment, then, as the time goes, his familiarity with this assignment improves. In addition, the knowledge of the behavior and the factors that usually affect this assignment increase. These two facts make it easier to deal with such assignment and to provide less effort in the future. The improvement in the familiarity of this assignment and the increase in the knowledge of its behavior occur due to the change in the behaviorism of this individual towards the assignment because of experience. In practice, a trainee usually learns and improves his skills while “On the Job Training”. As he becomes more proficient, his performance improves due to the learning effect. Moreover, even when a qualified operator is involved, “learning” often

occurs when he must apply new procedures or meets an unfamiliar situation. Conversely, forgetting has its influence once a large time is elapsed and if the individual stops performing the same assignment, then he may lose some of the experience that he gained in the past. In our study, we will limit the investigation on the learning and forgetting phenomena only to include areas where learning and forgetting are practicable in manufacturing systems, to determine the optimal production lot size. The traditional models for determining the optimal production lot size are based on the assumption that the production rate is constant. This in turn implies that quantities produced in a certain time are constant. If learning is to be considered, then such assumption is true only when the system reaches a steady state situation. On the other hand, when the operator is in the early period of the acquainted with, manufacturing a new product, dealing with new equipments, restarts after some delay , or even reestablishing new procedures, then this assumption becomes invalid and hence learning cannot be disregarded. Learning implies that the performance of a system engaged in a repetitive task improves with time. This improvement of the system (which may be a person or an organization consists of a group of persons but all aiming to accomplish a specific task) can be observed in manufacturing companies as a reduction in the cost and /or the time required to produce each successive unit. Factors usually inherent to this improvement may include the more effective use of tools and machines, increased familiarity with operational tasks and work environment, and enhanced management efficiency. These

factors can be reflected on the manufacturing system as a reduction in cost and /or time of production. On the other hand, a break in production will have an adverse effect on the cost and /or the time required to produce the unit. For example, it is reasonable to assume that if a large amount of time has elapsed between successive production runs, then, the production rate at the resumption of production might not be as high as when the production stopped. Hence, the cost and / or the time required to produce the first unit in the next production run will increase with the length of the break. This loss of performance over the production break is due to the forgetting phenomenon. If the effects due to learning or forgetting on savings in cost and / or time are significant, then the effects of learning and forgetting on the inventory system may also be significant. This will be the main subject of our thesis.

The thesis is organized as follows:

We started with the literature review, and then in chapter 1, we introduced some basic inventory models without learning. Model buildings and all mathematical formulations that guide to the optimal quantity were derived. Wright's learning curve and the results obtained in his study as well as the presence of the three conditions of learning that might exist are presented in chapter 2. We also reviewed and commented on some recent studies concerning the effect of full transmission of learning on the optimal production quantity for single item. Corrections as well as the justifications of the shortcomings that we felt appear in these models are also given in this chapter. The general case in which bounded learning is considered and

revisions of some recent models related to the effect of bounded learning on the optimal production quantity are treated in chapter 3. Viewpoint together with the corrections of the shortcomings that appear in these models are also included in this chapter. In chapter 4, we shall present the general production lot-sizing inventory model for deteriorating items with time varying demand and deterioration rates, shortages are allowed and partially backordered with exponential rate. A closed form for the total associated cost as well as rigorous mathematical methods that lead to a minimum total cost of the underlying inventory system are derived. In chapter 5, we reviewed some recent studies concerning the effect of learning and forgetting on the optimal production quantity for single item. The general production lot-sizing inventory model for deteriorating items with time varying demand and deterioration rates under the effect of learning and forgetting is introduced in this chapter. A closed form for the total relevant costs is derived, and rigorous mathematical methods that guide to a minimum total cost of the underlying inventory system are introduced. An illustrative example, which explains the applications of the theoretical results as well as its numerical verification are also given. Viewpoint that explains the shortcomings in previous studies is also included in this chapter.

## Literature review

The “ learning curve” introduced by Wright [27], was the first attempt to link the performance in a specific task to the number of times that task is repeated. When developing the curve, which shows variation of labor cost with production quantity, he showed that as production accumulates, the unit production time decreases by a constant percentage each time the cumulative quantity produced in doubles. Keachi and Fontana [19] provided an equation for the optimal lot size with learning, but did not provide methods for solving that equation. Steedman [24] extended upon the work of Keachie and Fontana by investigating the properties of their equation’s solution. Wortham and Mayyasi [26] provided a classical square root formula, restricted to low average holding cost items. Muth and Spermann [20] extended the above three works [19, 24, 26] by providing a numerical solution as well as an approximation for their transcendental cost function equation to determine the optimal lot size. Spradlin and Pierce [23] described lot sizing production runs whose production rate can be described by a learning curve. Their solution included the possibility of producing all units in one lot, or allowing for variable lot sizes. They used regression methods to determine the parameters of the learning and forgetting curves. Hancock [10] and Hoffman [11] made similar assumptions of those of Spradlin and Pierce [23]. Carlson [5] referred to the loss of learning in production as the amount of lost time in the start up of the production process. Sule [25], who based his analysis on the

forgetting curve presented by Carlson and Rowe [6], observed a subsequent drop in the production rate at the beginning of the next production cycle. He indicated that such production rate depends on the length of the interruption and the production rate when an intermittent occurs. Adler and Nanda [1] analyzed the effect of learning on production lot sizing models for both the single and multiple product cases. A general equation was developed for the average production time per unit when producing the annual demand in batches with some percentage of learning not retained between lots. They also developed a technique for estimating the drop in labor productivity as a decrease in the average production rate, because of production breaks. Two models for the economic production lot size were developed. The first model was restricted to equal lot sizes, and the second model was restricted to equal production intervals. Elmaghraby [7] reviewed and commented on two previously proposed models, corrected some minor errors in them and expanded one of them to accommodate a finite horizon. These models are of Spradlin and Pierce [23], and Sule [25]. He also suggested a different forgetting model from that of Carlson and Rowe [6], which he claimed that it is more consistent with the learning-forgetting relationship. Then he applied his suggestion to determine the optimal number and size of the lots in the finite and infinite horizon cases. Neither Elmaghraby [7] nor Sule [25] state how they selected their forgetting rates (slopes). Globerson et al [9], who based their studies on an experimental data, indicated that the degree of forgetting is a function of the break length and the level of experience gained

prior to the break. A power curve was identified as a proper forgetting model to depict the relationship between break length, performance time before the break, and the degree of forgetting. All the above authors assumed that the production rate is significantly higher than the demand rate. As a result, the inventory level during production period increases in an approximately linear fashion. Salameh et al [22] presented a modified production lot-sizing inventory model that incorporates the effect of full transmission of learning, which may be achieved during production period. They treated the infinite horizon case where production rate is not significantly higher than the demand rate, in which inventory level during production period raises in an increasingly nonlinear rate. Jaber and Salameh [17] attempted to generalize the work of Salameh et al [22] by assuming shortages to be allowed and backordered. In their study, they have defined the average production rate as a function of the amount produced. Further, they have stated that the average production rate can be approximated by dividing the accumulated production quantity, by the production time. Jaber and Bonney [12] extended the work of Salameh et al [22] by developing a mathematical model that describes the learning-forgetting relationship, referred to as the learning-forgetting curve model (LFCM). The (LFCM) was tested and shown to be consistent with the model presented by Globerson et al [9] with less than 1% deviation. With the (LFCM), it is possible to determine the value of the forgetting rate once its mathematical form is assumed. Jaber and Bonney [12] assumed that the forgetting slope is dependent on three factors. These factors

are the equivalent accumulated output of continuous production at the point of interruption, the minimum break to which the manufacturer assumes total forgetting, and the learning slope. They showed that forgetting had an adverse effect because of the drop in labor productivity. Jaber and Bonney [15] developed three models for the infinite and finite planning horizon. They showed that under partial transmission of learning the optimal policy was to carry fewer inventories in later lots. Jaber and Bonney [14] studied the effect of learning and forgetting on the optimal manufactured quantity with the consideration of intracycle, within cycle, backorders. They showed that the presence of interruptions results in longer cycle runs causing further increase on labor and inventory costs. Jaber and Bonney [16] categorized publications on learning curve effects on the lot-sizing problem into two groups. The first group includes those authors who studied the effects of learning on the economic manufacturing quantity (EMQ) with the assumption of instantaneous delivery. The second group includes those authors who studied the effects of learning on the (EMQ) with the assumption of finite delivery rate. Balkhi [4] proposed a general production lot sizing inventory model, which was dependent on his two previously proposed models [2, 3]. Then he generalized the work of Salameh et al [22] by incorporating learning effect in his general production lot size inventory model where each of the production, demand, and deterioration rates are general functions of time. Shortages are allowed but are partially backordered. This was done by defining the production rate as the number of units produced per unit time. The learning

curve presented by Wright implies that as cumulative output increases, the unit cost and /or the time required to produce each successive unit decreases. Furthermore, it gives the implication of a continuous reduction, rather than reaching some point, where the unit cost and /or the time required to produce each successive unit plateaus. To overcome the inadequacy in Wright's model, De Jong [18] suggested the bounded power function formulation by adding a new factor called the incompressibility factor to Wright's model. Fisk and Ballou [8] studied the manufacturing lot size problem under both the bounded and the unbounded learning situations. It was necessary in the unbounded learning case to resort to a numerical technique to find the optimal lot size for each given value of time. Then for different values of lot sizes, a numerical integration procedure was used, in particular Simpson's rule. Jaber and Bonney [13] attempted to present an efficient approximation for the closed algebraic form of the total inventory holding cost expression presented by Fisk and Ballou [8] by modifying the inventory variation outlined by Salameh et al [22]. Zhou [28] pointed out the errors in Jaber and Bonney's [13] mathematical formulation and proposed the correct model. He showed that the decrease in the optimum production quantity and the required production time are more significant in the second cycle than in later ones, due to the increase of labor productivity because of learning. Zhou and Lau [29] attempted to generalize the work of Zhou [28] by the consideration of shortages to be allowed and backordered.

## Chapter 1

### Some Basic inventory models without learning

#### 1.1 Introduction

The problem of inventory is dependent on the type of inventory system that the organization is adopting. To facilitate solving the problem of inventory, it is required to build models, which explain the inventory fluctuation. In this chapter, we shall introduce some basic inventory models, which may give a good insight of inventory problems and give an introduction for the models that will be given in subsequent chapters.

#### 1.2 Economic order quantity (EOQ)

In some cases such as retailer, wholesaler / distributor, where items are purchased externally, if the problem of inventory exists, then there are two main questions, which generally arise and face any organization. These are how many to order and when to order. Having too much inventory reduces both purchase and /or ordering costs, but it may tie up capital, which may lead to unnecessary holding cost and possibility of deteriorating items. Whereas having too little inventory reduces the holding cost, but it can result

in lost of customers, which may affect the reliability of the organization.

Answering these two questions will lead to the optimal level of inventory for any organization, which minimizes its total inventory cost.

Inventory costs, which are related to the operation of an inventory system, are caused by the actions or lack of actions that the organization is establishing.

The most common costs to an inventory system may include:

The purchase cost of an item obtained from an external source.

The order cost of issuing a purchase order to an outside source.

The holding /carrying cost for keeping items in storage.

The deterioration cost of items that expire, deteriorate, or break while in storage.

The shortage cost of an internal stock out of inventory.

As the total relevant cost of an inventory system is relatively high, the aim of inventory management is to minimize this total cost. However, it has to keep inventory at such a level that the goals and the organizational objectives are achieved.

The order size, which minimizes the total inventory cost, is known as the economic order quantity (EOQ). It is one of the most commonly used inventory models. The typical EOQ inventory model is shown in Figure 1-1, where  $q$  is the order size,  $B$  is the reorder point,  $r$  is the demand rate,  $a$  is the time where the order is placed, and  $b$  is the time at which a new order is received.

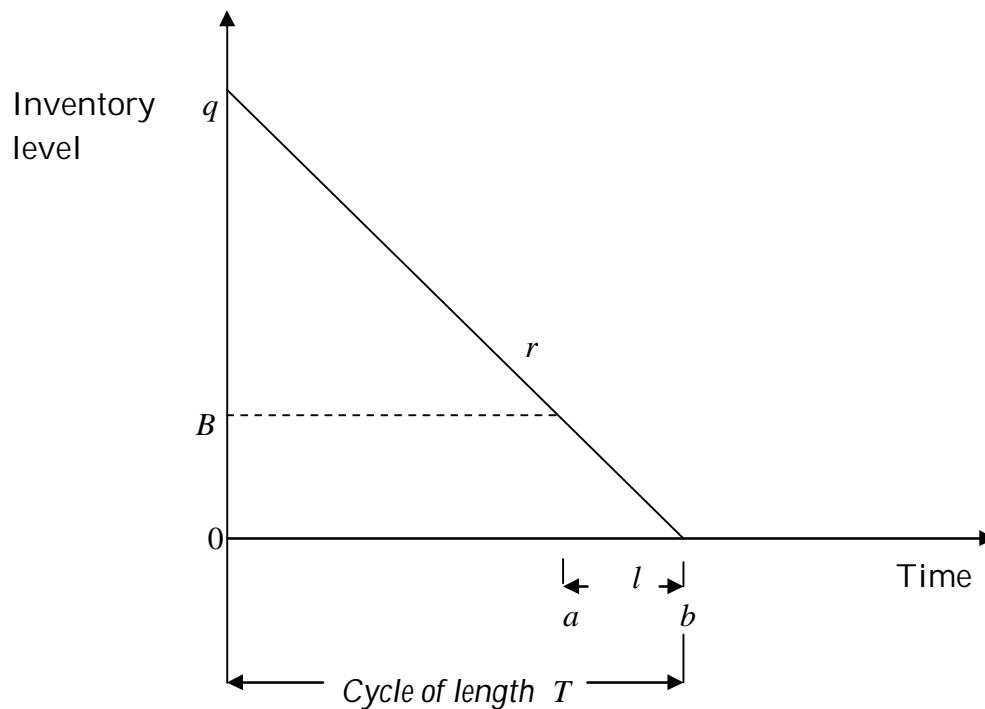


Figure 1-1. A typical inventory variation of an (EOQ) model in one cycle.

The inventory system operates as follows. When an order is received, the inventory level is  $q$  units. Thus, the maximum inventory level is  $q$  units. Inventory level declines continuously at a constant demand rate  $r$ , which is indicated by the negative sloping line. Once the inventory level reaches the reorder point  $B$ , a new order is placed for  $q$  units. The entire order is received and placed into inventory after a certain time  $l$ , ( $l$  is called the lead-time, which is the time that elapsed between the place of an order and its receipt). Once the inventory is depleted, a new order of  $q$  units is received.

### 1.2.1 Model formulation

Since the inventory level varies between a minimum of zero and a maximum of  $q$  units, the average inventory ( $AI$ ) in a cycle is given by

$$AI = \frac{(q+0)}{2} = \frac{q}{2}$$

The purchase cost during each cycle of length  $T$  is the purchase cost of an item say ( $c$ ) times the demand rate ( $r$ ), times  $T$ . The order cost during each cycle is the cost for placing an order say ( $k$ ). The holding cost during each cycle is the holding cost per unit per unit time say ( $h$ ) times  $T$  times the average inventory  $\frac{q}{2}$ . Therefore, the sum of the three cost components (purchase, order, and holding) is the total inventory cost during each cycle, i.e.

Total cost = purchase cost + order cost + holding cost, or

$$TC(q) = c q + k + h T \frac{q}{2} \quad (1-1)$$

Moreover, the total cost per unit time  $TCU$  is equal to  $\frac{TC(q)}{T}$

Or

$$TCU(q) = \frac{cq}{T} + \frac{k}{T} + \frac{hq}{2}$$

But,  $N = \frac{1}{T} = \text{number of orders per unit time}$  and  $r = \frac{q}{T}$ , so we have

$$TCU(q) = c r + \frac{kr}{q} + \frac{hq}{2} \quad (1-2)$$

The necessary condition for having a minimum is

$$\frac{dTCU(q)}{dq} = 0 \Leftrightarrow -\frac{kr}{q^2} + \frac{h}{2} = 0$$

which, after solving for  $q$ , yields the (EOQ) formula

$$\frac{kr}{q^2} = \frac{h}{2} \Rightarrow q^2 = \frac{2kr}{h} \Rightarrow$$

$$q^* = \sqrt{\frac{2kr}{h}} = \text{EOQ} \quad (1-3)$$

Noting that  $\frac{d^2TCU(q)}{dq^2} = \frac{2kr}{q^3} > 0, \forall q > 0,$

we conclude that the value  $q^*$  is the unique global minimum of  $TCU(q)$ .

To determine the reorder point, we recall that the demand is constant at a rate of  $r$  units. Therefore, the total demand during lead-time of  $l$  time units, where  $l < T^*$  is simply given by

$$B = rl \quad (1-4)$$

Hence, if an order is placed when the inventory level is  $B = rl$ , the order will arrive precisely when the inventory is depleted.

Note that if the lead-time  $l = 0$ , then this corresponds to an instantaneous delivery.

The minimum total cost per unit time is obtained by substituting  $q^*$  for  $q$  in the total inventory cost equation that is

$$TCU(q^*) = cr + \frac{kr}{q^*} + \frac{hq^*}{2}$$

where  $q^*$  is given by (1-3). Now, with a bit of algebra we can easily find

$$TCU(q^*) = cr + hq^*$$

### 1.3 Backordering (Shortages)

If an item required by a customer is not currently available in the organization stores, then the customer either goes to another place (a lost customer), or alternatively, places a backorder for the item. Some organizations are either sole supplier, providing a competitive price, or offering a discount for delaying the delivery of certain items. If this is the case, an organization does not lose the sale when its inventory is depleted. Instead, the customer has to wait for his order to be filled whenever a new order arrives. Therefore, backordering or shortages are the demand that will be filled some time later than desired.

Figure 1-2 depicts a typical inventory model in which shortages are allowed to occur at a constant demand rate  $r$  during time  $t_2$ . A maximum shortage level of  $x$  units is assumed. Once the inventory level reaches the reorder point  $B$ , an order is placed for  $q$  units. When the lot size or the order is received, the maximum shortage of  $x$  units is filled immediately, and the rest of the lot size is placed into inventory. Thus, the maximum inventory level is  $(q - x)$  units.

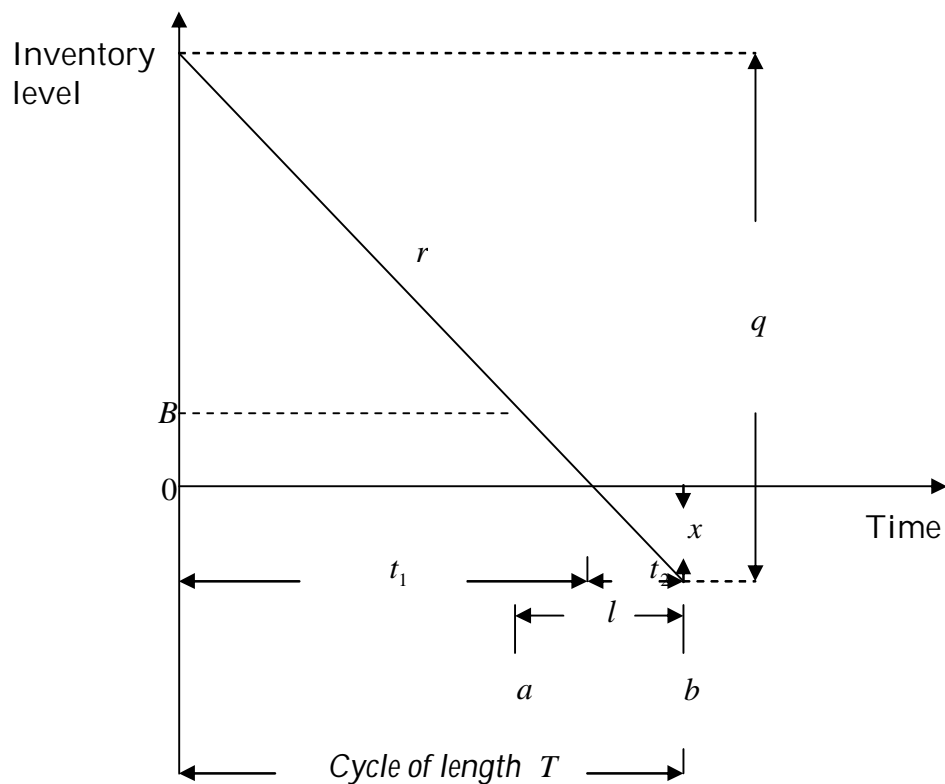


Figure 1-2. A typical inventory variation of an (EOQ) model in one cycle with shortages.

### 1.3.1 Model formulation

As in the no shortages case, a positive inventory of  $(q-x)$  units occurs during time  $t_1$ , so the average inventory (AI) is given by

$$AI = \frac{(q-x)}{2}$$

By keeping the notations of the previous model, then, the holding cost during time  $t_1$  is given by

$$\text{Holding cost} = \frac{h(q-x)t_1}{2}$$

Noting that  $t_1 = \frac{(q-x)}{r}$ , then, the holding cost during time  $t_1$  is given by

$$\text{Holding cost} = \frac{h(q-x)^2}{2r}$$

Shortage or negative inventory occurs during time  $t_2$ , so the average shortage inventory ( $AS$ ) during time  $t_2$  is given by

$$AS = \frac{(0+x)}{2} = \frac{x}{2}$$

Noting that  $t_2 = \frac{x}{r}$ , thus, the shortage cost during time  $t_2$  is given by

$$\text{Shortage cost} = \frac{sxt_2}{2} = \frac{sx^2}{2r}$$

where  $s$  is the shortage cost per unit per unit time.

The total inventory cost for each cycle of length  $T$ , ( $T = t_1 + t_2$ ) is given by

Total cost = purchase cost + order cost + holding cost + shortage cost, or

$$TC(q, x) = cq + k + \frac{h(q-x)^2}{2r} + \frac{sx^2}{2r} \quad (1-5)$$

Recalling that  $N = r/q = 1/T$ , then, the per unit time total cost is

$$TCU(q, x) = cr + \frac{kr}{q} + \frac{h(q-x)^2}{2q} + \frac{sx^2}{2q} \quad (1-6)$$

The necessary conditions for having a minimum are

$$\frac{\partial TCU(q, x)}{\partial q} = 0 \quad \& \quad \frac{\partial TCU(q, x)}{\partial x} = 0$$

Now, with a bit of algebra we can easily find

$$\frac{\partial TCU(q,x)}{\partial q} = -\frac{kr}{q^2} + \frac{2h(q-x)2q - 2h(q-x)^2}{4q^2} - \frac{2sx^2}{4q^2} = 0$$

$$\Leftrightarrow -\frac{kr}{q^2} + h - \frac{h}{2} - \frac{hx^2}{2q^2} - \frac{sx^2}{2q^2} = 0$$

which, after solving for  $q$ , gives

$$q^* = x\sqrt{\frac{(h+s)}{h}} + \sqrt{\frac{2kr}{h}} \quad (1-7)$$

Now,

$$\frac{\partial TCU(q,x)}{\partial x} = -\frac{4qh(q-x)}{4q^2} + \frac{4qsx}{4q^2} = 0$$

$$\Leftrightarrow \frac{-4q^2h + 4qhx}{4q^2} + \frac{sx}{q} = 0$$

which, after solving for  $x$ , gives

$$x^* = \frac{q^*h}{h+s} \quad (1-8)$$

Substituting  $x^*$  in E.q (1-7), we obtain

$$q^* = \frac{q^*h}{h+s}\sqrt{\frac{h+s}{h}} + \sqrt{\frac{2kr}{h}}, \text{ or}$$

$$q^* = \frac{\sqrt{h+s}}{\sqrt{s}}\sqrt{\frac{2kr}{h}} \quad (1-9)$$

Noting that

$$A = \frac{\partial^2 TCU(q,x)}{\partial q^2} = \frac{2kr}{q^3} + \frac{2hx^2}{q^3} + \frac{2sx^2}{q^3} > 0, \quad \forall q > 0, \forall x > 0,$$

$$L = \frac{\partial TCU(q,x)}{\partial x^2} = \frac{h}{q} + \frac{s}{q^2} > 0, \quad \forall q > 0,$$

$$E = \frac{\partial TCU(q, x)}{\partial q \partial x} = -\frac{x(h+s)}{q^2} < 0, \quad \forall q > 0, \forall x > 0, \text{ and}$$

$$E^2 = \frac{x^2(h+s)^2}{q^4} > 0, \quad \forall q > 0, \forall x > 0.$$

Then the Hessian matrix is given by

$$H(q, x) = \begin{bmatrix} A & E \\ E & L \end{bmatrix}$$

The principle minors of order 1 and 2 are  $h_{11} = A > 0$ , and it is easy to show that  $h_{22} = A \times L - E^2 > 0$ . Hence,  $H(q, x)$  is positive definite  $\Rightarrow TCU(q, x)$  is strictly convex, we conclude that the pair  $(q^*, x^*)$  forms the unique global minimum of  $TCU(q, x)$ .

Now, from (1-8) & (1-9), the minimum per unit time total cost is

$$TCU(q^*, x^*) = c r + \frac{kr}{q^*} + \frac{h(q^* - x^*)^2}{2q^*} + \frac{s(x^*)^2}{2q^*}$$

But from (1-8)  $q^* - x^* = \frac{q^* s}{h+s}$ . Substituting in the last  $TCU$  we obtain

$$TCU(q^*, x^*) = c r + \frac{kr}{q^*} + \frac{h\left(\frac{q^* s}{h+s}\right)^2}{2q^*} + \frac{s(x^*)^2}{2q^*}$$

which can be reduced, after a bit of algebra, to

$$TCU(q^*, x^*) = c r + \frac{q^* sh}{h+s} \quad (1-10)$$

To determine the reorder point, we note that, the demand is of constant rate of  $r$  units. Also, note that when the lot size or the order is received, the maximum shortage of  $x^*$  units is filled immediately. Therefore, the total demand during lead-time of  $l$  time units, where  $l < T^*$  is given by

$$B = rl - x^*$$

Hence, if an order is placed when the inventory level is  $B = rl - x^*$ , the order will arrive precisely when the inventory reaches the maximum shortage level of  $x^*$  units.

Note that if the lead-time  $l = 0$ , then this corresponds to an instantaneous delivery. In this case, the order is placed when a maximum shortage level of  $x^*$  units is reached, where then the placed order is received instantaneously.

#### 1.4 Economic production quantity (EPQ)

If the items are to be manufactured internally, then the problem of inventory in the manufacturing systems increases in magnitude and complexity. The problem of inventory exists because production and consumption are difficult to manage, since in most cases production and consumption differ in the rates so, either they provide or require stock. Thus, in manufacturing systems especially in batch-type production systems, where units are often produced and added to inventory in lot sizes (batches), it is required to determine the optimum number of units to be produced in each production run (each inventory cycle) so as to minimize total inventory cost. The economic production quantity formulation assumes gradual additions to stock over the production time. With this assumption, the inventory level is always less than the lot size, since production and consumptions occur simultaneously during the production time. However, the production cannot continue forever. Rather there are in general two stages, production & consumption stage, and pure consumption stage. The goal of inventory management is to minimize the total inventory cost and to satisfy the decision-making objectives. Having too much inventory, though it reduces set up costs, it may tie up capital, which may lead to unnecessary holding cost and possibility of deteriorating items. On the other hand too little inventory, even if it reduces the holding cost, it can result in lost customers or interrupted production. Therefore, there

is an optimal production lot size for any organization, which minimizes its total inventory cost.

To build up the EPQ model we shall below redefine  $c$  as the production cost and  $k$  as the set up cost and keep all other notations, which have been used in the above EOQ models with the addition that  $p(p > r)$  is the production rate (which is equal to the number of units produced per unit time).

Figure 1-3 depicts a typical inventory cycle in which shortages are allowed to occur.

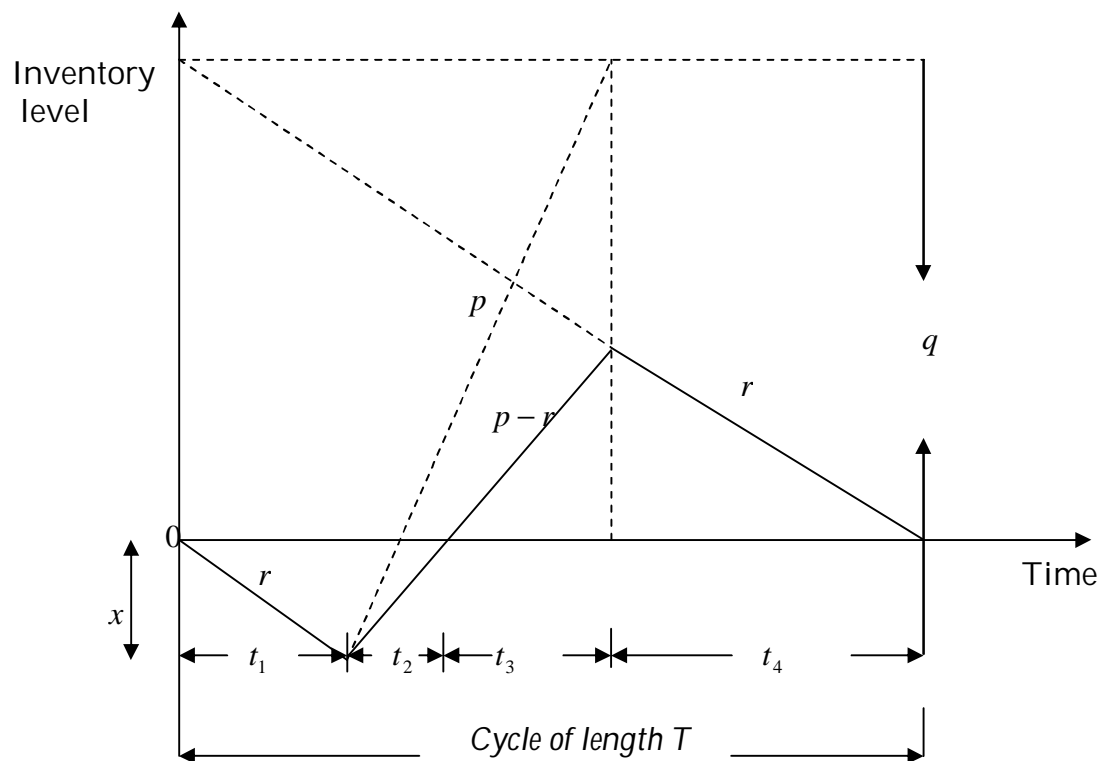


Figure 1-3. A typical inventory variation of an (EPQ) model in one cycle with shortages.

The system operates as follows. It starts at time  $t = 0$  at a demand rate  $r$  up to time  $t = t_1$  to allow shortages of  $x$  units to occur. Then production starts where the inventory level increases at a rate  $p - r$  in order to satisfy the demand and to eliminate the entire shortages of  $x$  units, where the inventory level becomes zero by time  $t_1 + t_2$ . At this time, the inventory level starts to go up with a rate  $p - r$  until time  $t_1 + t_2 + t_3$ , where production ceases and the inventory level reaches its maximum. Then the inventory level declines continuously at a rate  $r$  and becomes zero at time  $t_1 + t_2 + t_3 + t_4$  (the end of the cycle).

Now, the cost's components consist of shortage, holding, setup, and production costs.

#### 1.4.1 Model formulation

Note that shortage or negative inventory occurs during time  $t_1 + t_2$ .

The average shortage inventory ( $AS$ ) during time  $t_1$  is given by

$$AS = \frac{(0 + x)}{2} = \frac{x}{2}$$

But,  $t_1 = \frac{x}{r}$ , thus the shortage cost during time  $t_1$  is given by

$$\text{Shortage cost} = \frac{s x t_1}{2} = \frac{s x^2}{2r}$$

The average shortage inventory ( $AS$ ) during time  $t_2$  is given by

$$AS = \frac{(0+x)}{2} = \frac{x}{2}$$

Note that  $pt_2 = rt_1 + rt_2 \Leftrightarrow t_2 = \frac{x}{p-r}$ . Thus, the shortage cost during time  $t_2$  is

given by

$$\text{Shortage cost} = \frac{sxt_2}{2} = \frac{sx^2}{2(p-r)}$$

where  $s$  is the shortage cost per unit per unit time.

Hence, the shortage cost during time  $t_1 + t_2$  is given by

$$\begin{aligned} \text{Shortage cost} &= \frac{sx^2}{2} \left\{ \frac{1}{r} + \frac{1}{p-r} \right\} \\ &= \frac{sx^2}{2} \left\{ \frac{p-r+r}{r(p-r)} \right\} = \frac{sx^2 p}{2r(p-r)} \quad (1-11) \end{aligned}$$

We also note that positive inventory occurs during time  $t_3 + t_4$ , or during time  $T - (t_1 + t_2)$ , where,  $T = t_1 + t_2 + t_3 + t_4$ . Since the average inventory ( $AI$ ) during time  $t_3 + t_4$  varies between a minimum of zero and a maximum of  $(p-r)t_3$  units, the average inventory ( $AI$ ) during time  $t_3 + t_4$  is given by

$$AI = \frac{(p-r)t_3}{2}$$

But the production phase occurs during time  $t_2 + t_3$ , therefore

$$t_2 + t_3 = \frac{q}{p} \Leftrightarrow t_3 = \frac{q}{p} - t_2$$

where  $q$  is the (EPQ), recalling that  $t_2 = \frac{x}{p-r}$ , then

$$t_3 = \frac{q}{p} - \frac{x}{p-r} \Leftrightarrow t_3 = \frac{q(p-r) - xp}{p(p-r)}$$

Noting that  $T = \frac{q}{r}$ . In addition, the time  $t_1 + t_2$  is equal to  $\frac{x p}{r(p-r)}$

Therefore,

$$T - (t_1 + t_2) = \frac{q}{r} - \frac{x p}{r(p-r)}$$

$$\Leftrightarrow T - (t_1 + t_2) = \frac{q(p-r) - xp}{r(p-r)}$$

Thus, the holding cost during time  $T - (t_1 + t_2)$  is given by

$$\text{Holding cost} = \frac{h(p-r)t_3 [T - (t_1 + t_2)]}{2}$$

$$= h(p-r) \left\{ \frac{q(p-r) - xp}{2p(p-r)} \right\} \left\{ \frac{q(p-r) - xp}{r(p-r)} \right\}$$

$$= \frac{h [q(p-r) - xp]^2}{2pr(p-r)} \quad (1-12)$$

The total inventory cost for a cycle of length  $T$ , is given by

Total cost = production cost + setup cost + holding cost + shortage cost

which is, by the above results, given by

$$TC(q, x) = c q + k + \frac{h [q(p-r) - xp]^2}{2pr(p-r)} + \frac{s x^2 p}{2r(p-r)} \quad (1-13)$$

Recalling that  $N = 1/T = r/q$ , the per unit time total relevant cost is given by

$$TCU(q, x) = c r + \frac{kr}{q} + \frac{h [q(p-r) - xp]^2}{2qp(p-r)} + \frac{s x^2 p}{2q(p-r)} \quad (1-14)$$

The necessary conditions for having a minimum are

$$\frac{\partial TCU(q, x)}{\partial q} = 0 \quad \& \quad \frac{\partial TCU(q, x)}{\partial x} = 0$$

Now,

$$\begin{aligned} \frac{\partial TCU(q,x)}{\partial q} &= -\frac{kr}{q^2} + \frac{2h[q(p-r)-xp]2qp(p-r)^2}{4q^2p^2(p-r)^2} \\ &\quad - \frac{h[q(p-r)-xp]^2 2p(p-r)}{4q^2p^2(p-r)^2} - \frac{sx^2p}{2q^2(p-r)} = 0 \\ \Leftrightarrow \frac{\partial TCU(q,x)}{\partial q} &= -\frac{kr}{q^2} + \frac{h(p-r)}{2p} - \frac{x^2p}{2q^2(p-r)}(h+s) = 0 \end{aligned}$$

which, after solving for  $q$ , gives

$$\begin{aligned} q^2 &= \frac{x^2p^2(h+s)}{h(p-r)^2} + \frac{2kr}{h(p-r)} \\ \Leftrightarrow q^* &= \sqrt{\frac{(h+s)}{h} \frac{xp}{(p-r)}} + \sqrt{\frac{2krp}{h(p-r)}} \end{aligned} \quad (1-15)$$

Now,

$$\frac{\partial TCU(q,x)}{\partial x} = -h + \frac{hxp}{q(p-r)} + \frac{sxp}{q(p-r)} = 0$$

which, after solving for  $x$ , gives

$$x^* = \frac{q^*h(p-r)}{p(h+s)} \quad (1-16)$$

Substituting  $x^*$  in E.q (1-13), we obtain

$$\begin{aligned} q^* &= \frac{\sqrt{h}q^*}{\sqrt{h+s}} + \sqrt{\frac{2krp}{h(p-r)}}, \text{ or} \\ q^* &= \frac{\sqrt{h+s}}{\sqrt{s}} \sqrt{\frac{2krp}{h(p-r)}} \end{aligned} \quad (1-17)$$

Noting that

$$A = \frac{\partial^2 TCU(q,x)}{\partial q^2} = \frac{2kr}{q^3} + \frac{2px^2(h+s)}{q^3(p-r)} > 0, \quad \forall q > 0, \forall x > 0,$$

$$L = \frac{\partial TCU(q, x)}{\partial x^2} = \frac{p(h+s)}{q(p-r)} > 0, \quad \forall q > 0,$$

$$E = \frac{\partial TCU(q, x)}{\partial q \partial x} = -\frac{xp(h+s)}{q^2(p-r)} < 0, \quad \forall q > 0, \forall x > 0, \text{ and}$$

$$E^2 = \frac{x^2 p^2 (h+s)^2}{q^4 (p-r)^2} > 0, \quad \forall q > 0, \forall x > 0.$$

Then the Hessian matrix is given by

$$H(q, x) = \begin{bmatrix} A & E \\ E & L \end{bmatrix}$$

The principle minors of order 1 and 2 are  $h_{11} = A > 0$ , and it is easy to show that  $h_{22} = A \times L - E^2 > 0$ . Hence,  $H(q, x)$  is positive definite  $\Rightarrow TCU(q, x)$  is strictly convex, we conclude that the pair  $(q^*, x^*)$  forms the unique global minimum of  $TCU(q, x)$ .

Now, from (1-14) & (1-15), the minimum total cost per unit time is

$$TCU(q^*, x^*) = cr + \frac{kr}{q^*} + \frac{h[q^*(p-r) - x^*p]^2}{2q^*p(p-r)} + \frac{s(x^*)^2 p}{2q^*(p-r)}$$

But from (1-14),  $q^*(p-r) - x^*p = \frac{q^*s(p-r)}{(h+s)}$ . Substituting in the last  $TCU$

we obtain

$$TCU(q^*, x^*) = cr + \frac{kr}{q^*} + \frac{h \left[ \frac{q^*s(p-r)}{(h+s)} \right]^2}{2q^*p(p-r)} + \frac{s(x^*)^2 p}{2q^*(p-r)}$$

which can be reduced, after a bit of algebra, to

$$TCU(q^*, x^*) = cr + \frac{q^*sh(p-r)}{p(h+s)} \quad (1-18)$$

## 1.5 Conclusion

This chapter included some basic inventory models without learning, going from easy to difficult. The differences in inventory systems associated with all required definitions and explanations are introduced. Model buildings and all mathematical formulations that guide to the optimal order or produced quantity, which will lead to the minimum of the per unit time total inventory cost were derived. Even though, the above-discussed models are not our main subject of study, but, they have been introduced to show the significance of learning when a system reaches a steady state situation, and for the cases where they may be needed to clarify some areas, which might appear in later discussion.

## Chapter 2

### Some inventory models under learning effect

#### 2.1 Introduction

The “Learning Phenomenon” introduced by Wright [27], to study factors affecting the cost of airplanes was the first attempt to link the performance in a specific task to the number of times that task is repeated, resulted in the pioneer theory of the “learning curve”. Among these factors was the number of direct labor hours. As a result, when developing the curve, which shows variation of labor cost with production quantity, he showed that as production accumulates, the unit production time decreases by a constant percentage, e.g. 80 %, each time the cumulative quantity produced in doubles. Wright's power function formulation (which was based on an empirical data) is expressed as

$$T_n = T_1 n^{-b} \quad (2-1)$$

where  $T_n$  is the time required to produce the  $n^{th}$  unit,  $T_1$  is the time required to produce the first unit,  $n$  is the production count, and  $b$  is the slope of the learning curve, computed as  $b = -\frac{\log \lambda}{\log 2}$ , where  $\lambda$  is the learning rate.

Figure 2-1 depicts the decrease in the unit production time.

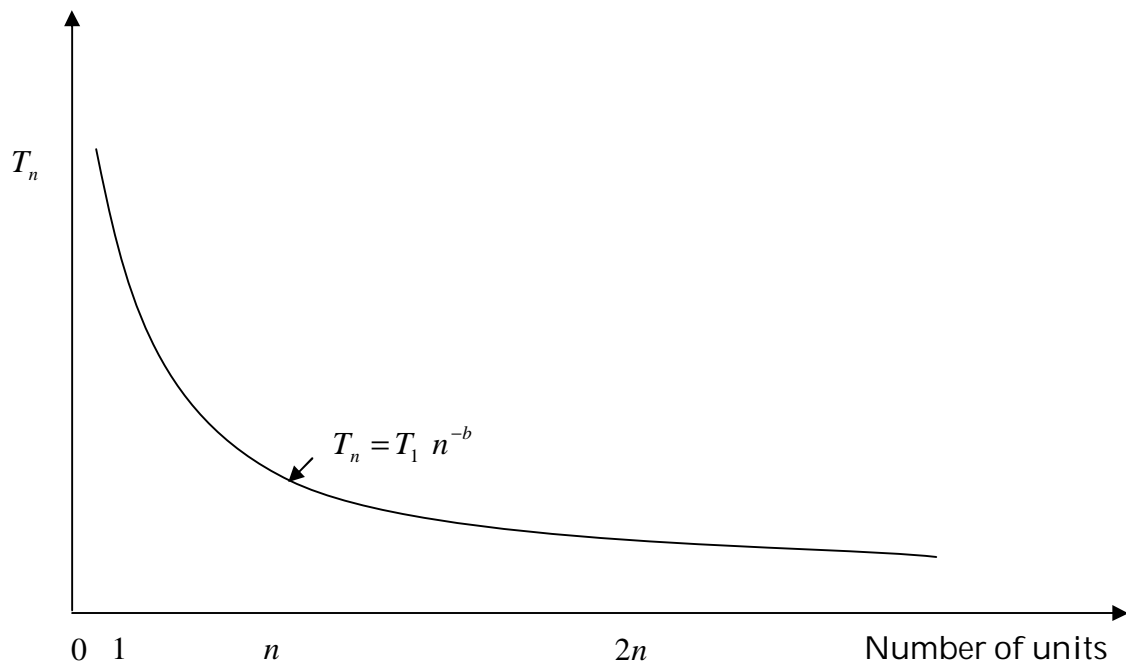


Figure 2-1. Wright's learning curve.

The plotted curve shows directly the relationship between the number of airplanes manufactured and the required labor cost and /or time. When the curve is plotted on log-log paper, it becomes a straight line. In Wright's study,  $b$  received the value of 0.32 when the curve was fitted to a set of data points, which express  $T_n$  as a function of  $n$ .

From Figure 2-1, it is obvious that,  $T_n > T_{2n}$ . In addition, since each time the cumulative quantity produced in doubles, the unit production time decreases by a constant percentage that is

$$\lambda = \frac{T_{2n}}{T_n} = \frac{T_1 (2n)^{-b}}{T_1 n^{-b}} = 2^{-b}$$

$$\Rightarrow b = -\frac{\log \lambda}{\log 2}, \text{ where } \lambda \text{ is the learning rate}$$

The logarithmic transformation of E.q (2-1) is

$$\log(T_n) = \log(T_1) - b \log(n)$$

This is the equation of a straight line with slope  $-b$ .

The value of  $b$  may be obtained by a linear regression using the logarithmic transformation of empirical data sets which express  $T_n$  as a function of  $n$ .

Substituting  $b = 0.32$  as in Wright's study

$$\Rightarrow \lambda = 2^{-b} = 2^{-0.32} = 0.80$$

The last result means that, the required labor cost for manufacturing the  $2n^{\text{th}}$  airplane is only 80% of the required labor cost for manufacturing the  $n^{\text{th}}$  airplane. The learning curve with a slope of 80 percent was identified from then on as a standard curve and is usually known as 80 percent learning curve. Wright's learning curve may describe group as well as individual performance. Moreover, organizational improvement is a kind of learning. Thus, as the learning curve describes the increasing skill of an individual by repetition of a specific task, likewise it may describe a more complex organism.

## 2.2 Transmission of learning

If learning is to be considered in a manufacturing system then, there are three situations of learning that might exist. These situations are:

### 2.2.1 There is full transfer of learning from period to period.

In this case, the same techniques, tools, individuals, procedures, work place, facilities, and so forth, are used in a repetitive manner for each production period. This means that the intermittent production will be treated as a continuous one, and a full transmission of learning does happen from one period to the next. This implies a reduction in the cost and / or the time required for producing next units in the same period as well as in next periods.

### 2.2.2 There is no transmission of learning from period to period.

In this case, completely new techniques, tools, individuals, procedures, work place, facilities, and so forth are used in the next production period. In this case, the cost and / or the time required to produce the first unit must be reestablished. Otherwise, the previous experience gained is totally forgotten. This means that the cost and / or the time required to produce the first unit remains the same at the beginning of each production run.

2.2.3 There is partial transmission of learning from period to period.

In this case, some of the previous techniques, tools, individuals, procedures, work place, facilities, and so forth are combined with the new allied to be used together in the next production period. This case seems to be the most important one, since in manufacturing systems, it is often the case that there exist some improvements in the general operations and procedures implementing by each system.

Next, we shall review some recent studies concerning the effect of learning on the optimal production quantity for single item when full transmission of learning does occur within each production period as well as in next periods.

### 2.3 Salameh et al's model

Salameh et al [22] presented a modified production lot-sizing inventory model that incorporates the effect of full transmission of learning, which may be achieved during production period. They treated the infinite horizon case where production rate is not significantly higher than the demand rate, in which inventory level during production period raises in an increasingly nonlinear rate. In particular, they treated the problem presented by Fisk and Ballou [8]. Figure 2-2 depicts the inventory level under this assumption.

We shall keep below all notations, which have been used up to now, with the addition that,  $d_m$  is the material cost, and  $\gamma$  is the labor cost per unit time as defined by Salameh et al [22].

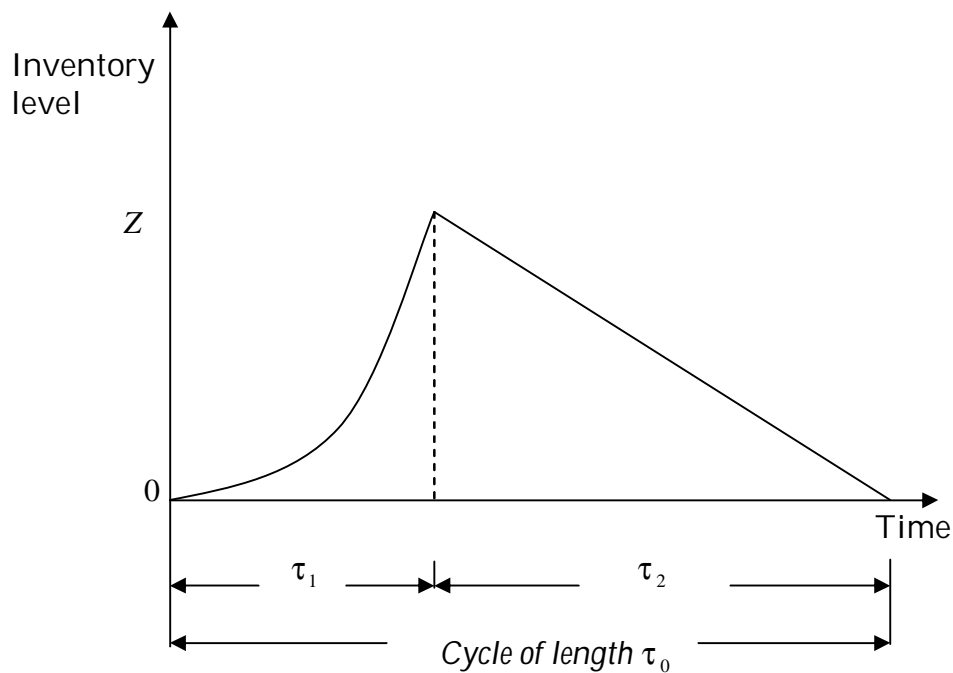


Figure 2-2. Inventory variation of an (EPQ) model in one cycle under learning effect.

Salameh et al [22] expressed the inventory level as a function of time as follows:

$$\phi(t) = q - r t \quad , \quad 0 \leq t \leq \tau_1 \quad (2-2)$$

$$\phi(t) = -r t + r \tau_0 \quad , \quad \tau_1 \leq t \leq \tau_2, \tau_0 = \tau_1 + \tau_2$$

where  $q$  is the total quantity produced during the production period  $\tau_1$ , and  $\tau_2$  is the pure consumption period. They assumed that the learning curve presented by Wright would be applicable for their model.

### 2.3.1 The mathematical model

The expected cumulative time to produce  $q$  units is given by

$$t(q) = t_1 + t_2 + t_3 + \dots + t_q = T_1 + T_1 2^{-b} + T_1 3^{-b} + \dots + T_1 q^{-b} = T_1 \sum_{j=1}^q j^{-b} \quad (2-3)$$

An approximation can be used, if we treated E.q (2-3) as a continuous function rather than a discrete one. Then E.q (2-3) can be rewritten as

$$t = t(q) = T_1 \sum_{j=1}^q j^{-b} \approx T_1 \int_0^q j^{-b} dj = \frac{T_1 q^{1-b}}{1-b} \quad (2-4)$$

which, after solving for  $q$ , gives

$$q = \left[ \frac{1-b}{T_1} t \right]^{\frac{1}{1-b}}, \quad 0 \leq t \leq \tau_1 \quad (2-5)$$

Substituting E.q (2-5) into E.q (2-2), then, the inventory level during the production period is given by

$$\phi(t) = \left[ \frac{1-b}{T_1} t \right]^{\frac{1}{1-b}} - r t, \quad 0 \leq t \leq \tau_1$$

The average inventory level with learning (*AIL*) for each cycle is given by

$$AIL = \int_0^{\tau_0} \phi(t) dt = \int_0^{\tau_1} \left[ \left( \frac{1-b}{T_1} t \right)^{\frac{1}{1-b}} - r t \right] dt + \frac{Z\tau_2}{2} \quad (2-6)$$

where  $Z$  is the maximum stock accumulated during the production period.

E.q (2-6) can be reduced, after a bit of algebra, to

$$AIL = \frac{q^2}{2r} - \frac{T_1 q^{2-b}}{(1-b)(2-b)} \quad (2-7)$$

The total inventory cost for a cycle of length  $\tau_0$ , is given by

Total cost (of producing  $q$  units) = labor cost + material cost + holding cost  
+ setup cost, or

$$TC(q) = \frac{\gamma T_1 q^{1-b}}{1-b} + d_m q + h \left[ \frac{q^2}{2r} - \frac{T_1 q^{2-b}}{(1-b)(2-b)} \right] + k \quad (2-8)$$

The total cost per unit time  $TCU$  is equal to  $\frac{TC(q)}{\tau_0} = \frac{TC(q)r}{q}$

Or

$$TCU(q) = \frac{\gamma r T_1 q^{-b}}{1-b} + d_m r + h \left[ \frac{q}{2} - \frac{r T_1 q^{1-b}}{(1-b)(2-b)} \right] + \frac{k r}{q} \quad (2-9)$$

The necessary condition for having a minimum is

$$\frac{dTCU(q)}{dq} = 0 \quad (2-10)$$

to show the convexity of the total cost function the first and second derivatives are obtained as shown below

$$\frac{dTCU(q)}{dq} = -b \frac{\gamma r T_1 q^{-b-1}}{1-b} h \left[ \frac{1}{2} - \frac{r T_1 q^{-b}}{(2-b)} \right] - \frac{k r}{q^2}$$

$$\frac{d^2TCU(q)}{dq^2} = b(b+1) \frac{\gamma r T_1 q^{-b-2}}{1-b} - h \frac{br T_1 q^{-b-1}}{(2-b)} + \frac{2kr}{q^3} > 0, \quad \forall q > 0$$

The Newton-Raphson method was used to solve the nonlinear, algebraic total cost model expressed in E.q (2-9). To illustrate the solution procedure, an inventory situation with the following parameters was considered by Salameh et al [22]:

Time to produce the first unit  $T_1 = 0.0625 \text{ days}$

Consumption rate  $r = 12 \text{ units / day}$

Direct material cost  $d_m = \$100 / \text{unit}$

Labor cost  $\gamma = \$10 / \text{hr}$

Learning slope  $b = 0.1$

Holding cost  $h = 0.2\$ \text{ unit / day}$

Fixed setup cost  $k = \$200$

By applying the Newton-Raphson method the two equations (2-9) and (2-10), were used to obtain the optimum production quantity in the first cycle,

yielding  $q^* = 216$ . The time required to produce the first unit in the second cycle, namely unit number (217), is given by  $T_{217} = T_1 (217)^{-0.1} = 0.0365 \text{ days}$ .

The same procedures were used all over again to obtain the optimum production quantity in the second cycle, to give  $q^* = 184$ . The optimum production quantity levels off after the seventh cycle. In their study, they have treated the cycles independently from one another in order to simplify the mathematical formulation. It is to be noted in this model, that the decrease in the optimum production quantity, the required production time, and the increase in the maximum stock accumulated are more significant in the second cycle than in later ones. This is because of the increase of labor productivity due to learning. They also showed that the time required for producing the first unit decreases with each additional cycle, while the optimum production quantity decreases, and the maximum stock accumulated increases, with each additional cycle until they level off at the seventh cycle.

Table 2-1, gives the computation of the optimal lot size  $q$  for cycle 1, while table 2-2, gives the optimal values for the time required to produce the first unit, the optimum production quantities, the total time required to produce these quantities, and the maximum stock accumulated for nine successive cycles.

Table 2-1. Computation of optimal  $q$  for cycle 1

$q$	$T'(q)$	$T''(q)$
20	-5.99	0.60
30	-2.65	0.18
45	-1.17	0.05
66	-0.51	0.02
97	-0.21	0.01
136	-0.08	0.00
177	-0.03	0.00
206	-0.01	0.00
215	-0.00	0.00
216	-0.00	0.00
216	-0.00	0.00
216	-0.00	0.00

Table 2-2. Optimum quantities

Cycle	$T_1$	$q^*$	$t(q^*)$	$Z$
1	0.0625	216	8.750	111
2	0.0365	184	4.425	131
3	0.0343	182	4.118	132
4	0.0331	180	3.943	133
5	0.0322	180	3.822	134
6	0.0315	179	3.731	134
7	0.0310	178	3.657	135
8	0.0305	178	3.596	135
9	0.0301	178	3.544	135

## 2.4 Jaber and Salameh's model

Jaber and Salameh [14] attempted to generalize the work of Salameh et al [22] by assuming shortages to be allowed and completely backordered. In their study, they have defined the average production rate  $\hat{p}(q)$  as a function of the amount  $q$  produced as follows:

Jaber and Salameh [14] stated that, if the amount ordered is  $q$  units, and if these are to be consumed at a rate of  $r$  units per unit time, and if  $p$  units are made each unit of time during the production run, then the average inventory level ( $AI$ ) will be

$$AI = \frac{q(p-r)}{2p} = \frac{q(1-r/p)}{2} \quad (2-11)$$

Recalling that the average inventory level with learning ( $AIL$ ) as given by Salameh et al [22] is

$$AIL = \frac{q}{2} - \frac{T_1 q^{1-b} r}{(1-b)(2-b)} \quad (2-12)$$

Then they provided an estimation of  $\hat{p}(q)$  by equating Eqs (2-11) and (2-12) yielding

$$\hat{p}(q) = \frac{(1-b)(2-b)}{2T_1} q^b \cong \frac{(1-b)}{T_1} q^b \quad (2-13)$$

Further, they have stated that the average production rate  $\hat{p}(q)$  can be approximated by dividing the accumulated production quantity  $q$  by the production time  $t(q)$  given by E.q (2-4). Figure 2-3 depicts the inventory model treated by Jaber and Salameh [14].

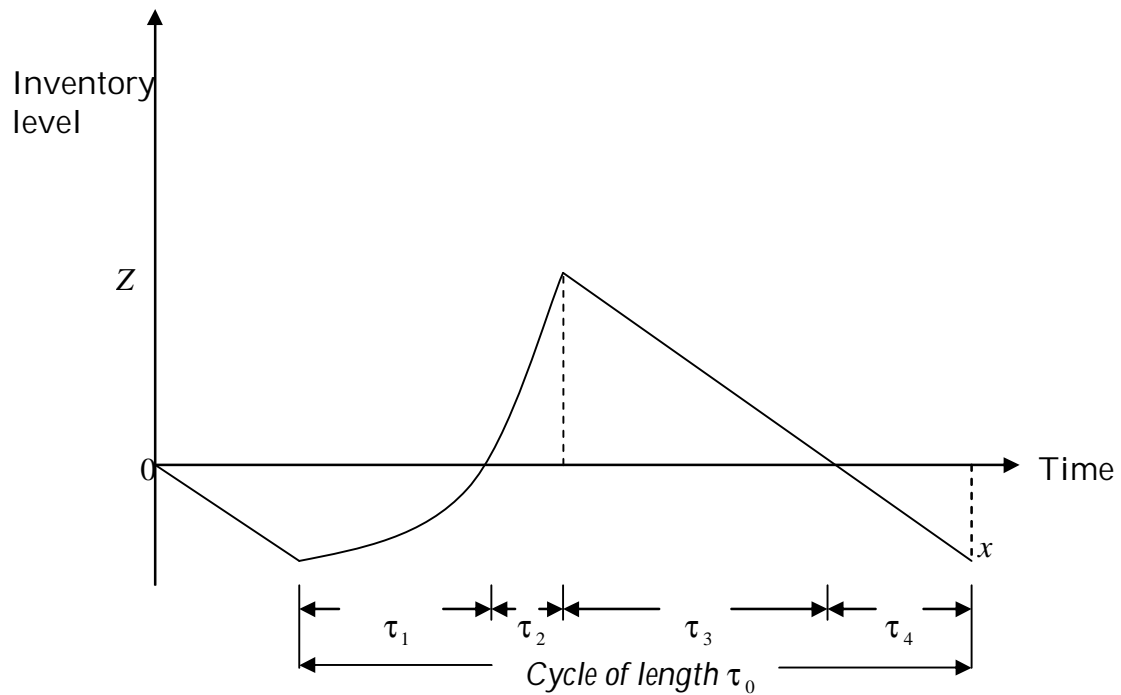


Figure 2-3. Inventory variation of an (EPQ) model in one cycle under learning effect with shortages.

#### 2.4.1 The mathematical model

The shortage or negative inventory occurs during time  $\tau_1 + \tau_4$ , so, the average shortage inventory (*ASL*) is given by

$$ASL = \frac{x^2 \hat{p}(q)}{2r(\hat{p}(q) - r)} \quad (2-14)$$

where the period of negative inventory,  $\tau_1 + \tau_4$  is given by

$$\tau_1 + \tau_4 = \frac{x \hat{p}(q)}{r(\hat{p}(q) - r)}$$

However, they did not show how they obtain this value.

The period of positive inventory is given by

$$\tau_2 + \tau_3 = \frac{q}{r} - \frac{x \hat{p}(q)}{r(\hat{p}(q) - r)}$$

The maximum inventory accumulated during time  $\tau_1 + \tau_2$  is given by

$$Z = q - x - \frac{rT_1q^{1-b}}{1-b}$$

Therefore, the average inventory is

$$AIL = \left\{ \frac{q}{2} - \frac{x \hat{p}(q)}{2(\hat{p}(q) - r)} \right\} \left\{ \frac{q}{r} - \frac{x}{r} - \frac{T_1q^{1-b}}{1-b} \right\}$$

The total inventory costs for a cycle of length  $\tau_0$ , is given by

Total cost = labor cost + material cost + holding cost + setup cost + shortages cost, i.e.

$$TC(q, x) = \frac{\gamma T_1 q^{1-b}}{1-b} + d_m q + h \left\{ \frac{q}{2} - \frac{x \hat{p}(q)}{2(\hat{p}(q) - r)} \right\} \left\{ \frac{q}{r} - \frac{x}{r} - \frac{T_1 q^{1-b}}{1-b} \right\} + k + \frac{s x^2 \hat{p}(q)}{2r(\hat{p}(q) - r)} \quad (2-15)$$

The total cost per unit time  $TCU$  is equal to  $\frac{TC(q, x)}{\tau_0} = \frac{TC(q, x)r}{q}$

Or

$$TCU(q, x) = \frac{\gamma T_1 q^{-b} r}{1-b} + d_m r + h \left\{ \frac{q}{2} - \frac{x \hat{p}(q)}{2(\hat{p}(q) - r)} \right\} \left\{ 1 - \frac{x}{q} - \frac{r T_1 q^{-b}}{1-b} \right\} + \frac{k r}{q} + \frac{s x^2 \hat{p}(q)}{2q(\hat{p}(q) - r)} \quad (2-16)$$

The necessary conditions for having a minimum are

$$\frac{\partial TCU(q, x)}{\partial q} = 0 \quad \& \quad \frac{\partial TCU(q, x)}{\partial x} = 0$$

Their undetectable determining of the value of  $x$ , in terms of  $q$  will give

$$x = \left[ 1 - \frac{r}{\hat{p}(q)} \right] \frac{hq}{(h + s)} \quad (2-17)$$

We shall below correct some of the above results.

### 2.5.1 Correction of Salameh et al's [22] model <sup>a</sup>

As in Salameh et al [22],

$$\phi(t) = q - rt \quad , \quad 0 \leq t \leq \tau_1 \Rightarrow \phi(0) = q - r \times 0 = q \neq 0$$

$$\phi(t) = -rt + r\tau_0 \quad , \quad \tau_1 \leq t \leq \tau_2, \tau_0 = \tau_1 + \tau_2 \Rightarrow \phi(\tau_2) = -r\tau_2 + q \neq 0.$$

Hence, both equations are invalid. On the other hand, during their calculation Salameh et al [22] have made a correct integral.

Taking into consideration that the inventory levels at times,  $\phi(\tau_0) = \phi(0) = 0$ , the correct expression of the inventory level for the above model as a function of time is given by

$$\phi(t) = q(t) - r t \quad , \quad 0 \leq t \leq \tau_1$$

$$\phi(t) = r\tau_0 - rt \quad , \quad \tau_1 \leq t \leq \tau_1 + \tau_2, \tau_0 = \tau_1 + \tau_2$$

Also Salameh et al [22] dropped the following constraint

$$\int_0^{\tau_1} \left[ \left( \frac{1-b}{T_1} t \right)^{\frac{1}{1-b}} - r t \right] dt = r\tau_2$$

which insures the fact that the inventory level must have an equal value for time  $t = \tau_1$ .

## 2.5.2 Correction of Jaber and Salameh's [14] model <sup>a</sup>

Recalling Figure 2-2, since units are withdrawn at a rate of  $(p-r)$  during production period  $\tau_1 = \frac{q}{p}$ , then the average inventory level ( $AI$ ) will be

$$AI = \frac{q(p-r)}{2p} = \frac{q(1-r/p)}{2}$$

Recalling that, the without learning, average shortage or negative inventory occurs during time  $t_1 + t_2$ , as given in E.q (1-11), is given by

$$ASL = \frac{x^2 p}{2r(p-r)}$$

This is the same value obtained in E.q (2-14), with respect to (w.r.t)  $\hat{p}(q)$ .

In addition, the maximum shortages without learning as derived in E.q (1-16), is given by

$$x^* = \frac{q^* h(p-r)}{p(h+s)}$$

This is the same value obtained in E.q (2-17), (w.r.t)  $\hat{p}(q)$ .

Even though, the expression of the inventory level as a function of time was not included in their paper, we believe that Jaber and Salameh [14] have used similar definition as in Salameh et al [22]. Moreover, they have treated the production rate during time  $\tau_1$  as linear rate rather than nonlinear one, and then they substituted the average production rate  $\hat{p}(q)$  to obtain E.qs (2-14) and (2-17), respectively. Further, the period of positive inventory is dependent on the average shortage level they have obtained. Their estimation, which is

provided in E.q (2-13), may be valid only if shortages are not considered. Besides the above mentioned shortcomings, Jaber and Salameh [14] dropped two constraints, the first one to insure the fact that the inventory level must have an equal value for time  $t = t_4$ , while the second one insures this fact for time  $t = t_2$ . Therefore, the whole model needs to be reformulated starting from the inventory level and the production rate in terms of time along with the two dropped constraints.

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<sup>a</sup> , These corrections are due to the author.

## 2.6 Balkhi's model

Balkhi [4] provided some mathematical techniques and algorithms, which expedited and explored many areas in solving inventory problems. Among these was his pioneer generalization of the production lot-sizing inventory model, which was dependent on his two previously proposed models [2, 3]. Then, using the clue mentioned by Jaber and Salameh [14], he defined the production rate  $P(t)$  in its more natural sense as

$$P(t) = \frac{\text{Number of units produced up to time } t}{t} \quad (2-18)$$

He also assumed that  $P(t)$  is subject to a full transmission of learning, and the initial production rate is assumed to be  $1/t_1$ , where  $t_1$  is the time required to produce the first unit. Further, he generalized the work of Salameh et al [22] by incorporating learning effect in his general production lot size inventory model where each of the production, demand, and deterioration rates are general functions of time. Shortages are allowed but are partially backordered.

To establish the general EPQ model we shall give below the following assumptions and notations as presented by Balkhi [4].

1. A single item is produced in batches at an increasing rate, denoted by  $P(t)$ .
2. The items are subject to deterioration when they are effectively in stock and there is no repair or replacement of deteriorated items.

3. The demand and deterioration rates are known functions of time denoted by  $D(t)$  and  $\delta(t)$  respectively.

4. Shortages are allowed, but only a fraction  $\beta$  ( $0 \leq \beta \leq 1$ ) is backordered and the rest  $(1 - \beta)$  is lost.

5. The cost parameters are as follows:

$c$  = Unit production cost, which includes materials, labors and manufacturing costs.

$h$  = Unit holding cost per unit per unit time.

$b$  = Unit shortage cost per unit per unit time for backordered items.

$s$  = Unit shortage cost per unit per unit time for lost items.

$k$  = Set up cost per set up.

He also defined the inventory level as a function of time as follows:

Let  $I(t)$  denote the inventory level at time  $t$ . The system starts operating at time  $T_0$ , in which shortages are first allowed to occur at a demand rate  $\beta D(t)$  up to time  $T_1$ , at which time a maximum backordered level of  $S$  units is reached. Then the system starts production and the inventory level increases at a rate  $P(t) - D(t)$  in order to fulfill the demand and to clear the entire shortages of  $S_j$  units, where the inventory level becomes zero by time  $T_2$ .

Now the inventory level starts to go up with a rate  $P(t) - D(t) - \delta(t)I(t)$  until time  $T_3$  where the inventory level reaches its maximum. Then the inventory level declines continuously at a rate  $D(t) - \delta(t)I(t)$  and becomes zero at time

$T_4$  (the end of the cycle). The process is repeated. Figure 2-4 depicts the general inventory model treated by Balkhi [4].

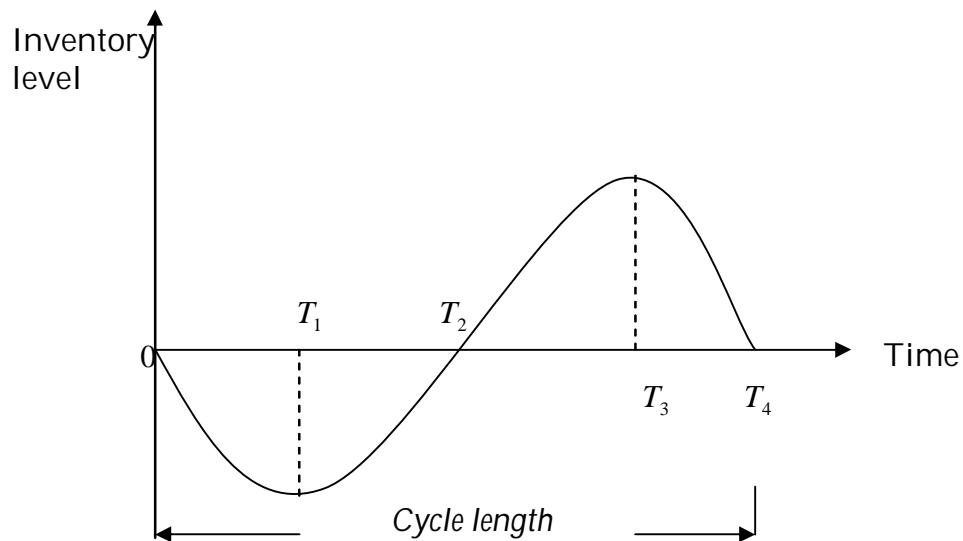


Figure 2- 4. Inventory variation of a general (EPQ) model in one cycle under learning effect with shortages.

### 2.6.1 Model formulation

The changes in the inventory level depicted in Figure 2-4, can be represented by the following differential equations

$$\frac{dI(t)}{dt} = -\beta D(t) \quad T_0 \leq t < T_1 \quad (2-19)$$

with the initial condition  $I(T_0) = 0$ ,

$$\frac{dI(t)}{dt} = -[P(t) - D(t)] \quad T_1 \leq t < T_2 \quad (2-20)$$

with the ending condition  $I(T_2) = 0$ ,

$$\frac{dI(t)}{dt} = P(t) - D(t) - \delta(t)I(t) \quad T_2 \leq t < T_3 \quad (2-21)$$

with the initial condition  $I(T_2) = 0$ , and

$$\frac{dI(t)}{dt} = -D(t) - \delta(t)I(t) \quad T_3 \leq t \leq T_4 \quad (2-22)$$

with the ending condition  $I(T_4) = 0$ .

Let  $g'(t) = \delta(t)$  and  $G'(t) = e^{-g(t)}$

Then the solutions of the above differential equations are

$$I(t) = -\beta \int_{T_0}^t D(u) du \quad T_0 \leq t < T_1 \quad (2-23)$$

$$I(t) = - \int_t^{T_2} [P(u) - D(u)] du \quad T_1 \leq t < T_2 \quad (2-24)$$

$$I(t) = e^{-g(t)} \int_{T_2}^t [P(u) - D(u)] e^{g(u)} du \quad T_2 \leq t < T_3 \quad (2-25)$$

$$I(t) = e^{-g(t)} \int_t^{T_4} D(u) e^{g(u)} du \quad T_3 \leq t \leq T_4 \quad (2-26)$$

respectively.

$$\text{Let } I(t_1, t_2) = \int_{t_1}^{t_2} I(u) du$$

then from (2-25) and (2-26), (using integration by parts) we can easily find

$$I(T_2, T_3) = \int_{T_2}^{T_3} [G(T_3) - G(u)] [P(u) - D(u)] e^{g(u)} du \quad (2-27)$$

$$I(T_3, T_4) = \int_{T_3}^{T_4} [G(u) - G(T_3)] D(u) e^{g(u)} du \quad (2-28)$$

Similarly (2-23) and (2-24) lead, respectively, to

$$I(T_0, T_1) = \int_{T_0}^{T_1} (T_1 - u) D(u) du \quad (2-29)$$

$$I(T_1, T_2) = \int_{T_1}^{T_2} [u - T_1] [P(u) - D(u)] du \quad (2-30)$$

Note that we can set  $T_0 = 0$  without loss of generality. The cost components for the given inventory system are as follows:

$$\text{Items cost} = c \int_{T_1}^{T_3} P(u) du .$$

Note that this cost includes the deterioration cost.

$$\text{Holding cost} = h [ I(T_2, T_3) + I(T_3, T_4) ]$$

$$\text{Shortage cost for backordered items} = b [ I(0, T_1) + I(T_1, T_2) ]$$

$$\text{Shortage cost for lost items} = s(1 - \beta) \int_0^{T_1} P(u) du$$

Thus, the total cost per unit time of the underlying inventory system during the cycle  $[0, T_4]$ , which consists of the production costs, inventory holding cost, shortage costs for the backordered and the lost items and set up cost, as a function of  $T_1, T_2, T_3$  and  $T_4$ , say  $W(T_1, T_2, T_3, T_4)$ , is given by

$$\begin{aligned}
W(T_1, T_2, T_3, T_4) = & \frac{1}{T_4} \left\{ c \int_{T_1}^{T_3} P(u) du + h \left[ \int_{T_2}^{T_3} \{G(T_3) - G(u)\} \{P(u) - D(u)\} \right. \right. \\
& \times e^{g(u)} du + \left. \int_{T_3}^{T_4} \{G(u) - G(T_3)\} D(u) e^{g(u)} du \right] \\
& + b \left[ \beta \int_0^{T_1} (T_1 - u) D(u) du + \int_{T_1}^{T_2} (u - T_1) (P(u) - D(u)) du \right] \\
& \left. + s (1 - \beta) \int_0^{T_1} D(u) du + k \right\} \quad (2-31)
\end{aligned}$$

His goal was to find  $T_1, T_2, T_3$  and  $T_4$  that minimize  $W(T_1, T_2, T_3, T_4)$ , where  $W(T_1, T_2, T_3, T_4)$  is given by (2-31). But, the variables  $T_1, T_2, T_3$  and  $T_4$  are related to each other through the following relations

$$0 < T_1 < T_2 < T_3 < T_4 \quad (2-32)$$

$$\beta \int_0^{T_1} D(u) du = \int_{T_1}^{T_2} [P(u) - D(u)] du \quad (2-33)$$

$$e^{-g(T_3)} \int_{T_2}^{T_3} [P(u) - D(u)] e^{g(u)} du = e^{-g(T_3)} \int_{T_3}^{T_4} D(u) e^{g(u)} du \quad (2-34)$$

Relation (2-32) expresses the natural monotonicity constraints, since otherwise the given problem, would have no meaning. Relations (2-33) and (2-34) ensure the fact that the inventory levels must have equal values for  $t = T_1$  and for  $t = T_3$  respectively. Thus, the goal was to solve the following optimization problem, which he called problem (p)

$$(p) = \begin{cases} \text{Minimize } W(T_1, T_2, T_3, T_4) & \text{subject to (2-32)} \\ h_1 = 0 \text{ \& } h_2 = 0 \end{cases}$$

where  $W(T_1, T_2, T_3, T_4)$  is given by (2-31) and  $h_1, h_2$  are respectively given by

$$h_1 = \beta \int_0^{T_1} D(u) du - \int_{T_1}^{T_2} [P(u) - D(u)] du$$

$$h_2 = \int_{T_2}^{T_3} [P(u) - D(u)] e^{g(u)} du - \int_{T_3}^{T_4} D(u) e^{g(u)} du$$

The solution procedure for this general model can be found in details in Balkhi [4], and sufficient conditions for the global optimality of the solutions to problem (p) can be established by quite similar methods to those been used in Balkhi [2, 3].

## 2.6.2 Model formulation under full transmission of learning

Since full transmission of learning implies a reduction in the time required for producing next units in the same period as well as in next periods, Balkhi [4] provided the following approximation for E.q (2-1)

$$t_{ij} = t_{1j} i^{-r} \quad (2-35)$$

where  $t_{ij}$  is the time required to produce the  $i^{th}$  unit in the  $j^{th}$  cycle and  $r$  is the slope of the learning curve.

Let  $t_j$  be the time required to produce  $i$  units in the  $j^{th}$  cycle then

$$t_j = \sum_{k=1}^i t_{1j} k^{-r} \approx t_{1j} \int_0^i k^{-r} dk = t_{1j} \frac{i^{1-r}}{1-r} \quad (2-36)$$

In addition, from the definition of  $P(t)$  it follows that

$$P(t_j) = \frac{i}{t_j} = \frac{1-r}{t_{1j} i^{-r}}$$

Now, let  $Q_j$  be the amount produced in the interval  $[T_{2j}, T_{3j}]$  and  $S_j$  as defined in 2.6. Then, from (2-36) we have

$$T_{3j} - T_{2j} = t_{1j} \frac{Q_j^{1-r}}{1-r} \quad (2-37)$$

Also,

$$S_j = \beta \int_0^{T_{1j}} D(u) du \quad (2-38)$$

Note that the right hand side (RHS) of (2-38) is an increasing function of  $T_{1j}$ , so  $T_{1j}$  can be uniquely determined as a function of  $S_j$ , say

$$T_{1j} = f_{1j}(S_j) \quad (2-39)$$

From (2-39) and (2-33),  $T_{2j}$  can be uniquely determined as a function of  $T_{1j}$ , hence of  $S_j$ , say

$$T_{2j} = f_{2j}(S_j) \quad (2-40)$$

Substituting (2-40) in (2-37) we find that  $T_{3j}$  can be uniquely determined as a function of  $S_j$  &  $Q_j$ , say

$$T_{3j} = f_{3j}(S_j, Q_j) \quad (2-41)$$

From which and (2-34),  $T_{4j}$  can be uniquely determined as a function of  $S_j$  &  $Q_j$ , say

$$T_{4j} = f_{4j}(S_j, Q_j) \quad (2-42)$$

Note that (2-40) through (2-42) have resulted from direct substituting of  $h_1 = 0$  and  $h_2 = 0$ . Substituting these results in (2-31), we obtain the following unconstrained problem with the variables  $S_j$  &  $Q_j$

$$\begin{aligned}
L(S_j, Q_j) = \frac{1}{f_4} \left\{ c \int_{f_1}^{f_3} P(u) du + h \left[ - \int_{f_2}^{f_3} G(u) \{P(u) - D(u)\} e^{g(u)} du \right. \right. \\
\left. \left. + \int_{f_3}^{f_4} G(u) D(u) e^{g(u)} du \right] \right. \\
\left. + b \left[ -\beta \int_0^{f_1} u D(u) du + \int_{f_1}^{f_2} u (P(u) - D(u)) du \right] \right. \\
\left. + s (1 - \beta) \int_0^{f_1} D(u) du + k \right\} \quad (2-43)
\end{aligned}$$

Letting  $l(S_j, Q_j) = \frac{L(S_j, Q_j)}{f_4} \Leftrightarrow l(S_j, Q_j) = f_4 L(S_j, Q_j)$

Then, the necessary conditions for having a minimum are

$$\frac{\partial L}{\partial S_j} = 0 \quad \text{and} \quad \frac{\partial L}{\partial Q_j} = 0 \quad (2-44)$$

Special methods first have been used to simplify the above optimization problem, and then a Nonlinear programming package has been used to determine the optimal values of  $T_{1j}, T_{2j}, T_{3j}$  and  $T_{4j}$ , which will lead to the optimal values of  $S_j$  &  $Q_j$  that guide to a minimum total cost function.

To illustrate the solution procedure, an inventory situation with the following parameters was considered by Balkhi [4]:

$$D(t) = 2at + d \quad t \geq 0 \quad \delta(t) = \frac{a_1}{b_1 - b_2 t} \quad , t \geq 0, \quad b_1 \geq a_1 \geq 0 \quad \text{and} \quad b_1 > b_2 \geq 0$$

The slope of the learning curve  $r = 0.075$

The time required to produce the first unit in the first cycle  $t_{11} = 0.0015 \text{ year}$

Percentage of backordered items  $\beta = 0.85$

Unit production cost  $c = \$50$

Unit shortage cost per year  $s = b = \$0.5$

Unit holding cost per year  $h = \$0.1$

Set-up cost per set-up  $k = \$200$

Parameters of demand rate  $a = 250 \text{ units/year}, d = 125 \text{ units/year}$

Parameters of deterioration rate  $a_1 = b_2 = 10 \text{ units/year}, b_1 = 1000 \text{ units/year}$

The parameter “ $a$ ” represents the rate of change in the demand rate.

The parameters  $a_1, b_1,$  and  $b_2$  are just function parameters so that  $a_1/b_1$  represents the deterioration rate at time  $t = 0$ .

Table 2-3 shows the optimal values of  $T_{1j}, T_{2j}, T_{3j}$  and  $T_{4j}$ , which will lead to the optimal values of  $S_j$  &  $Q_j$  and the corresponding total minimum cost for five successive cycles. Suppose that  $Q$  units to be produced in the interval  $[T_{1j}, T_{3j}]$ , then taken  $t_{11} = 0.0015 \text{ year}$ , in the first cycle, which results in a total number of  $Q = 134.35$  units. Then from (2-35) we found that the time required to produce the first unit in the second cycle, namely unit number 135.35, is equal to  $t_{135.35} = 0.0015(135.35)^{-0.075} = 0.001038$ . The same procedure is repeated for the other cycles. The tabulated results indicate that both the time required to produce the first unit and the total time required to produce the optimum production quantity decrease as the number of production runs increases. This is due to the increase in the production rate. It is worth noting here that this general model implies the previous discussed models. Furthermore, it overcomes all shortcomings corrected in this chapter.

Table 2-3: Optimal results under full transmission of learning for the illustrative example with  $r = 0.075$ .

Cycle no.	Time required to produce the first unit	No. of shortages units	No. of units produced in $T_1, T_3]$	Total no. of units produced	Time required to produce $Q$ units	Production rate	production start time	End of shortages period	End of Production period	Cycle end	Min total cost
$j$	$t_{1j}$	$S_j$	$Q_j$	$Q$			$T_{1j}$	$T_{2j}$	$T_{3j}$	$T_{4j}$	$L$
1	0.0015	81.37	29.55	134.35	0.150849	890.63	0.417398	0.537310	0.568247	0.89234	6784.4
2	0.001038	83.41	21.42	125.38	0.097843	1281.44	0.424561	0.506839	0.522494	0.87234	6427.5
3	0.000722	84.94	20.82	122.91	0.066876	1837.88	0.429875	0.486355	0.496751	0.85947	6379.6
4	0.000503	84.98	20.53	120.98	0.045914	2634.93	0.430012	0.468673	0.475926	0.84996	6332.2
5	0.000351	87.58	19.52	120.18	0.031816	3774.20	0.438846	0.465949	0.470765	0.84907	6283.6

## 2.7 Conclusion

In this chapter, we explained Wright's learning curve to include all parameters presented in his study and the meaning of the results obtained. The presence of the three conditions of learning that might exist is also discussed. We reviewed and commented on some of the recent studies concerning the effect of full transmission of learning on the optimal production quantity for single item. Corrections as well as the justifications of the shortcomings that we felt appear in these models are also given. It is to be noted here, that Salameh et al's [22] model and Jaber and Salameh's [14] model are special cases of that of Balkhi's [4]. Even though, many researchers have treated the learning phenomenon (see also Jaber and Bonney [16]), however, for the reasons of further generalization, and since the models that we have reviewed contain most of the previous work, we have limited the investigation only to cover these models. Now, in the case when the system is subject to a full transmission of learning, then intermittent productions are treated as continuous ones. Then, the potential question arises here, is that when would the learning cease?

One might think that even if production is continued forever, this would not cause any significance, once the unit cost and / or the time required to produce each successive unit attains its minimum, or reaches the standard cost and / or time, whether this stabilization is due to task complexity or no more learning could occur. On the other hand, the learning curve presented

by Wright implies that as cumulative output increases, the unit cost and / or the time required to produce each successive unit decreases. Moreover, it gives the implication of a continuous reduction, rather than reaching some point, where the unit cost and / or the time required to produce each successive unit plateaus. To overcome the inadequacy in Wright's model, for instance, if machines do some portions of the task or these portions are not subject to learning, then , such problem imposes a different learning curve from that of Wright's, which will be introduced in the following chapter.

## Chapter 3

### Some inventory models under bounded learning effect

#### 3.1 Introduction

A common theoretical drawback of Wright's learning curve  $T_n = T_1 n^{-b}$  (recall E.q (2-1)) is that for  $b > 0$  then  $T_n \rightarrow 0$  as  $n \rightarrow \infty$  which means that the time required to produce the  $n^{\text{th}}$  unit can be neglected as  $n$  takes on relatively large values. Such situation will also make the cumulative production approach to infinity when  $n$  increases. This, in fact, is unreasonable conclusion, since in real-world problems, it is often that, after a certain time of cumulative learning, in a production system the system reaches steady state situation, in which case  $T_n$  will approach an almost certain value. To avoid such a shortcoming in Wright's formula, De Jong [18] suggested the following bounded learning curve

$$T_n = T_1 m + (1 - m) T_1 n^{-b} \quad (3-1)$$

where  $m$  is a new factor called the incompressibility factor so that  $0 \leq m \leq 1$ . Note that for  $m = 0$  then  $T_n = T_1 n^{-b}$  which coincides with Wright's formula. For  $m = 1$  then  $T_n = T_1$  this means that there is no learning. However, for  $0 < m < 1$  we conclude from E.q (3-1) that for  $b > 0$  and as  $n \rightarrow \infty$ , then

$$T_n \rightarrow m T_1 \quad (3-2)$$

Note that relation (3-2) means that as the number of produced units increases, then the time required to produce the  $n^{\text{th}}$  unit approaches  $mT_1 < T_1$ , since  $0 < m < 1$  (recall that  $T_1$  is the time required to produce the first unit). Note that the above results overcome all shortcomings and / or drawbacks that might happen when using Wright's learning curve. Figure 3-1 depicts the behavior of the De Jong's bounded learning curve.

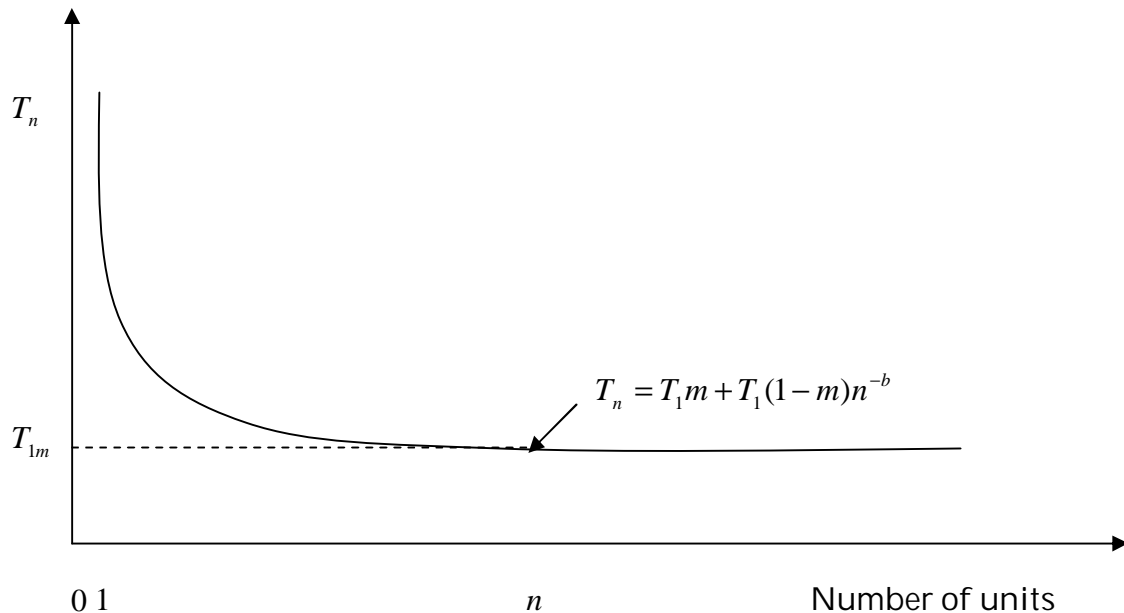


Figure 3-1. De Jong's bounded learning curve.

The mathematical complexity associated with the bounded expression presented in E.q (3-1), compelled most researchers to assume that learning curve conforms to the unbounded power function formulation presented by Wright. Fisk and Ballou [8] studied the manufacturing lot-size problem under

both the bounded and the unbounded learning situations. They represented the bounded power function formulation as follows

$$t_i = P_y + a_y i^{-b} \quad (3-3)$$

where  $t_i$  is the time required to produce the  $i^{th}$  unit in years,  $i$  is the production count,  $P_y$  is the fixed component of labor cost in years,  $a_y$  is the variable component of the time needed to produce the first unit in years, and  $b$  is the learning slope. The value of the incompressibility factor  $m$ , is estimated as  $m = P_y / (P_y + a_y)$ .

In order to solve the inventory problem outlined in Figure 2-1, Fisk and Ballou [8] defined  $Q(t)$ , as the number of units manufactured in time  $t$ ,  $0 < t < \tau_1$ . To find the expression for  $Q$ , it was necessary to solve the following equation

$$t = \int_{s+0.5}^{s+q+0.5} (P_y + a_y i^{-b}) di \quad (3-4)$$

for  $q$  as a function of time  $t$ , where  $s$  is the number of units produced prior to producing  $q$  units.

By carrying out the integration in (3-4), the production time given by Fisk and Ballou [8] was in the form

$$t = \tau_1 = \frac{a_y}{1-b} \left[ (q + s + 0.5)^{(1-b)} - (s + 0.5)^{(1-b)} \right] + P_y q \quad (3-5)$$

The above expression can not be solved for  $q$  as a function of time  $t$ , in a closed algebraic form. It was necessary in their case to resort to a numerical technique to find  $q$  for a given value of  $t$ . The value for  $Q(t)$  can be obtained

for each choice of  $t$ , using the Newton-Raphson numerical search method. Then for several values of  $Q$ , a numerical integration procedure was used, in particular Simpson's rule.

Next, we shall review some recent studies concerning the effect of bounded learning on the manufacturing lot-size problem.

### 3.2 Jaber and Bonney's model

Jaber and Bonney [13] attempted to present an efficient approximation for the closed algebraic form of the total inventory holding cost expression presented by Fisk and Ballou [8] as follows:

First, they modified the inventory variation outlined by Salameh et al [22] by joining the origin point  $O$  to the point  $A$ , as depicted in Figure 3 -2. Now, by keeping all notations as defined by Salameh et al [22], where  $\tau_{1i}$  is the cumulative time to produce a total quantity of  $q_i$  units in the  $i^{th}$  cycle, and  $\tau_{2i}$  is the pure consumption period. Then the integration of E.q (3-1) over the proper limits yields

$$\tau_{1i} = T_{11}mq_i + (1-m)T_{1i} \frac{q_i^{1-b}}{1-b} \quad (3-6)$$

where  $T_{1i}$  is the time required to produce the first unit in the  $i^{th}$  cycle, computed as  $T_{1i} = T_{11} (1 + \sum_{n=1}^{i-1} q_n)^{-b}$  where  $q_n$  is the quantity produced in production cycle  $n$ .

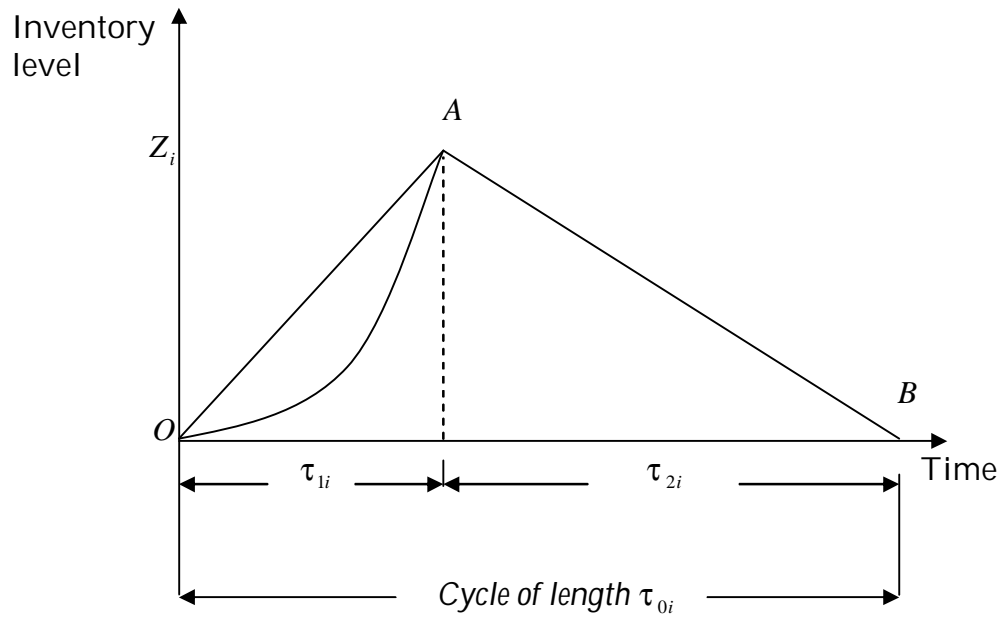


Figure 3-2. A modified inventory variation of an (EPQ) model in one cycle for the bounded learning case.

### 3.2.1 The mathematical model

Let  $Z_i = q_i - r\tau_{1i}$ , where  $r$  is the demand rate per unit time, then  $Z_i$  is the maximum stock accumulated during the production period  $\tau_{1i}$ . Jaber and Bonney [13] considered Figure 3-2 as two areas. Area( $OAB$ ), which forms their modification triangle, and area ( $O\dot{A}B$ ), which forms the tended triangle outlined by Salameh et al [22].

Now, the average inventory level over area ( $O\dot{A}B$ ) is given by

$$\text{Area } (O\dot{A}B) = \frac{q_i^2}{2r} - \frac{T_{1i}q_i^{2-b}}{(1-b)(2-b)} \quad (3-7)$$

This is the average inventory level with learning (*AIL*) as given by Salameh et al [22].

The average inventory level over area (*OAB*) is given by

$$\text{Area } (OAB) = \frac{q_i}{2r}(q_i - r\tau_{li}) \quad (3-8)$$

The difference in area between E.q (3-7) and (3-8) is given by

$$A(q_i) = \frac{bq_i\tau_{li}}{2(2-b)} \quad (3-9)$$

However, this value is incorrect.

Their presentation of the average inventory level with bounded learning (*AIBL*) for each cycle is given by

$$AIBL = \frac{q_i}{2r}(q_i - r\tau_{li}) - (1-m)\frac{bq_i\tau_{li}}{2(2-b)} \quad (3-10)$$

However, they did not justify how E.q (3-10) can be obtained.

If  $m$  equals zero or one, the (*AIBL*) will reduce to either E.q (3-7) or (3-8), respectively.

Substituting E.q (3-6) into E.q (3-10), the total inventory cost for a cycle of length  $\tau_{oi}$ , is given by

Total cost (of producing  $q_i$  units) = labor cost + material cost + holding cost + setup cost, i.e.

$$\begin{aligned}
TC(q_i) = & \gamma \left( T_{11} m q_i + (1-m)T_{1i} \frac{q_i^{1-b}}{1-b} \right) + d_m q_i \\
& + h \left\{ \frac{q_i}{2r} \left[ q_i - r \left( T_{11} m q_i + (1-m)T_{1i} \frac{q_i^{1-b}}{1-b} \right) \right] \right. \\
& \left. - (1-m) \frac{bq_i}{2(2-b)} \left( T_{11} m q_i + (1-m)T_{1i} \frac{q_i^{1-b}}{1-b} \right) \right\} + k \quad (3-11)
\end{aligned}$$

The total cost per unit time  $TCU$  is equal to  $\frac{TC(q_i)}{\tau_{0i}} = \frac{TC(q_i)r}{q_i}$

Or

$$\begin{aligned}
TCU(q_i) = & \gamma \left( T_{11} m r + (1-m)T_{1i} \frac{r q_i^{-b}}{1-b} \right) + d_m r \\
& + h \left\{ \frac{1}{2} \left[ q_i - r \left( T_{11} m q_i + (1-m)T_{1i} \frac{q_i^{1-b}}{1-b} \right) \right] \right. \\
& \left. - (1-m) \frac{br}{2(2-b)} \left( T_{11} m q_i + (1-m)T_{1i} \frac{q_i^{1-b}}{1-b} \right) \right\} + \frac{rk}{q_i} \quad (3-12)
\end{aligned}$$

The necessary condition for having a minimum is

$$\frac{dTCU(q_i)}{dq_i} = 0 \quad (3-13)$$

In their comparison with Fisk and Ballou [8], Jaber and Bonney [13] stated that the mathematical expression derived to determine the approximate total cost was given by substituting E.q (3-5) into E.q (3-12). The resulting total cost expression was then used to determine the optimum quantities as well as the total cost for the example presented by Fisk and Ballou [8]. However, they did

not show the resulting total cost expression. They also stated that the results obtained by using their model were shown to be consistent with those of Fisk and Ballou [8], with less than .5%.

Next, we shall do some corrections to the above shortcomings given in Jaber and Bonney [13].

Now, recalling Figure 3-2, the difference in area between area( $OAB$ ) and area( $O\dot{A}B$ ), as represented by Jaber and Bonney [13] is

$$A(q_i) = \frac{bq_i\tau_{il}}{2(2-b)}$$

The average inventory level over area( $O\dot{A}B$ ) is given by

$$\text{Area } (O\dot{A}B) = \frac{q_i^2}{2r} - \frac{T_{li}q_i^{2-b}}{(1-b)(2-b)}$$

This is the average inventory level with learning ( $AIL$ ) as given by Salameh et al [22].

The average inventory level over area ( $OAB$ ) is given by

$$\text{Area } (OAB) = \frac{q_i}{2r}(q_i - r\tau_{li})$$

The correct difference in area between E.q (3-7) and (3-8) is given by

$$A^*(q_i) = \frac{q_i}{2r}(q_i - r\tau_{li}) - \left\{ \frac{q_i^2}{2r} - \frac{T_{li}q_i^{2-b}}{(1-b)(2-b)} \right\}$$

$$A^*(q_i) = \frac{T_{li}q_i^{2-b}}{(1-b)(2-b)} - \frac{\tau_{li}q_i}{2} \quad (3-14)$$

### 3.3 Viewpoint <sup>a</sup>

We feel that the consideration of the two areas, area( $OAB$ ) and area ( $O\dot{A}B$ ) has no meaning, since area( $OAB$ ) is calculated for the without learning case, while area( $O\dot{A}B$ ) is calculated for the learning case. Also Jaber and Bonney's [13] mathematical formulation was incorrect starting from the first equation they have derived.

For area( $OAB$ ), recall that E.q (1-12), as derived in (chapter 1) was in the form

$$AI = \frac{[q(p-r) - xp]^2}{2pr(p-r)}$$

Substituting  $x = 0$  (since we do not have shortage) in the above equation, we obtain

$$AI = \frac{q^2(p-r)}{2pr}$$

$$AI = \frac{q}{2r} \left( q - \frac{qr}{p} \right) = \frac{q}{2r} (q - r\tau_{li})$$

This equation is valid only in the steady state situation. Hence, we cannot substitute  $\tau_{li}$  as in E.q (3-6) into the above  $AI$ , because this  $AI$  obtained using

$\tau_{li}$  as in the steady state situation, where then  $\tau_{li} = \frac{q}{p}$ . Similarly, area( $O\dot{A}B$ ) is

calculated for the learning case, hence, we cannot substitute  $\tau_{li}$  as in E.q (3-6)

into the  $AIL$  obtained for area( $O\dot{A}B$ ). Because the correct  $\tau_{li}$  in this case is

given by  $\tau_{li} = T_{li} \frac{q_i^{1-b}}{1-b}$ .

Also Jaber and Bonney's [13] did not include the constraint which insures the fact that the inventory level must have an equal value for time  $t = \tau_{li}$ .

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<sup>a</sup> , This viewpoint is due to the author.

### 3.4 Zhou's model

Besides the above pointed out errors, Zhou [28] pointed out some other errors in Jaber and Bonney's [13] mathematical formulation and proposed the correct model. This was done by extending the work of Salameh et al [22].

Now, by keeping all notations as defined above, Zhou [28] expressed the inventory level as a function of time as follows

$$I(t) = \begin{cases} y - r t & , 0 \leq t \leq \tau_{1i} \\ r (\tau_{0i} - t) & , \tau_{1i} \leq t \leq \tau_{0i}, \tau_{0i} = \tau_{1i} + \tau_{2i} \end{cases} \quad (3-15)$$

where  $y(t)$  is the produced quantity.

#### 3.4.1 Model formulation

If  $\tau_{1i}$  is the cumulative time to produce a total quantity of  $y$  units in the  $i^{th}$  cycle, then

$$t(y) = \int_0^y [ T_{11}m + T_{1i}(1-m)X^{-b} ] dX = T_{11}my + \frac{T_{1i}(1-m)y^{1-b}}{1-b} \quad (3-16)$$

Letting  $y = q_i$  in E.q (3-16), we obtain

$$\tau_{1i} = T_{11}mq_i + (1-m)T_{1i} \frac{q_i^{1-b}}{1-b}$$

Therefore, the average inventory level with bounded learning (*AIBL*) for the  $i^{th}$  cycle is given by

$$\begin{aligned}
AIBL &= \int_0^{\tau_{0i}} I(t)dt = \int_0^{\tau_{1i}} [y - r t] dt + \frac{Z_i \tau_{2i}}{2} \\
&= \int_0^{q_i} y [T_{1i} m + T_{1i} (1-m) y^{-b}] dy - r \int_0^{\tau_{1i}} t dt + \frac{(q_i - r \tau_{1i})^2}{2r} \\
&= \frac{T_{1i} m q_i^2}{2} + \frac{T_{1i} (1-m) q_i^{2-b}}{2-b} - r \frac{\tau_{1i}^2}{2} + \frac{q_i^2}{2r} - q_i \tau_{1i} + r \frac{\tau_{1i}^2}{2} \\
&= \frac{T_{1i} m q_i^2}{2} + \frac{T_{1i} (1-m) q_i^{2-b}}{2-b} + \frac{q_i^2}{2r} - T_{1i} m q_i^2 - \frac{T_{1i} (1-m) q_i^{2-b}}{1-b} \\
&= \frac{q_i^2}{2r} [1 - r T_{1i} m] + \frac{(1-b) T_{1i} (1-m) q_i^{2-b} - (2-b) T_{1i} (1-m) q_i^{2-b}}{(2-b)(1-b)} \\
&= \frac{q_i^2}{2r} [1 - r T_{1i} m] - \frac{T_{1i} (1-m) q_i^{2-b}}{(2-b)(1-b)} \tag{3-17}
\end{aligned}$$

Substituting E.q (3-17) into E.q (3-11), the total inventory cost for a cycle of length  $\tau_{0i}$ , is given by

Total cost (of producing  $q_i$  units) = labor cost + material cost + holding cost + setup cost, i.e.

$$\begin{aligned}
TC(q_i) &= \gamma \left( T_{1i} m q_i + (1-m) T_{1i} \frac{q_i^{1-b}}{1-b} \right) + d_m q_i \\
&\quad + h \left\{ \frac{q_i^2}{2r} [1 - r T_{1i} m] - \frac{T_{1i} (1-m) q_i^{2-b}}{(2-b)(1-b)} \right\} + k \tag{3-18}
\end{aligned}$$

The total cost per unit time  $TCU$  is equal to  $\frac{TC(q_i)}{\tau_{0i}} = \frac{TC(q_i)r}{q_i}$

Or

$$TCU(q_i) = \gamma \left( T_{11} m r + (1-m) T_{1i} \frac{r q_i^{-b}}{1-b} \right) + d_m r + h \left\{ \frac{q_i}{2} [1 - r T_{11} m] - \frac{r T_{1i} (1-m) q_i^{1-b}}{(2-b)(1-b)} \right\} + \frac{rk}{q_i} \quad (3-19)$$

The necessary condition for having a minimum is

$$\frac{dTCU(q_i)}{dq_i} = 0 \quad (3-20)$$

The Newton-Raphson method was used to solve the nonlinear, algebraic total cost model expressed in E.q (3-19). To show the solution procedure, an inventory system with the following parameters was considered by Zhou [28]:

Factor of incompressibility	$m = 0.25$
Time to produce the first unit	$T_{11} = 0.0625 \text{ days}$
Consumption rate	$r = 12 \text{ units / day}$
Direct material cost	$d_m = \$100 / \text{unit}$
Labor cost	$\gamma = \$80 / \text{hr}$
Learning slope	$b = 0.1$
Holding cost	$h = 0.2\$ \text{ unit / day}$
Fixed setup cost	$k = \$200$

By applying the Newton-Raphson method the two equations (3-19) and (3-20), were used to obtain the optimum production quantity in the first cycle, yielding  $q^* = 258$ . The time required to produce the first unit in the second

cycle, namely unit number (259), is given by  $T_{259} = T_{11}m + (1-m)T_{11}(259)^{-0.1} = 0.0425 \text{ days}$ . The same procedure was used all over again to obtain the optimum production quantity in the second cycle, to give  $q^* = 222$ . It is to be noted in this model, that the decrease in the optimum production quantity and the required production time are more significant in the second cycle than in later ones, due to the increase of labor productivity because of learning. Table 3 -1 gives the optimum values for the time required to produce the first unit, the optimum production quantities, the total time required to produce these quantities, and the corresponding minimum total cost for ten successive cycles.

Table 3 -1. Optimum quantities

Cycle	$T_{1i}$	$q_i^*$	$t(q_i^*)$	$ATC(q_i^*)$
1	0.0625	258	11.743	1264.22
2	0.0425	222	8.049	1257.87
3	0.0409	219	7.776	1257.33
4	0.0399	218	7.644	1257.03
5	0.0393	217	7.543	1256.82
6	0.0388	216	7.457	1256.64
7	0.0384	216	7.415	1256.51
8	0.0381	215	7.348	1256.41
9	0.0378	215	7.318	1256.30
10	0.0376	214	7.259	1256.22

Remark

It is to be noted here, that Zhou [28] did not include the following constraint in his model

$$\int_0^{\tau_{li}} [y - r t] dt = r\tau_{2i}$$

which coincides with the fact that the inventory level should have an equal value for time  $t = \tau_{li}$ .

### 3.5 Zhou and Lau's model

Zhou and Lau [29] attempted to generalize the work of Zhou [28] by the consideration of shortages to be allowed and fully backordered. Figure 3-3 depicts the inventory variation under this situation.

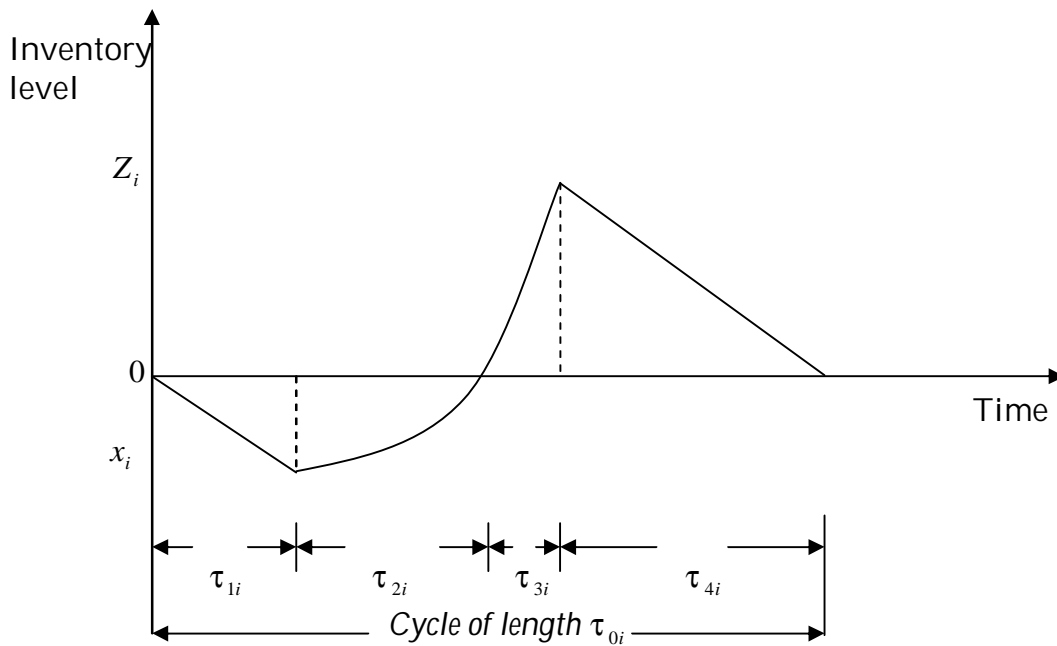


Figure 3-3. A typical inventory variation of an (EPQ) model in one cycle for the bounded learning case with shortages.

By keeping all notations as defined above, Zhou and Lau [29] expressed the inventory level as a function of time as follows:

The negative inventory level, say  $S(t)$  is given by

$$S(t) = \begin{cases} rt & , 0 \leq t \leq \tau_{1i} \\ x_i - y + r(t - \tau_{1i}) & , \tau_{1i} \leq t \leq \tau_{1i} + \tau_{2i} \\ 0 & , \tau_{1i} + \tau_{2i} \leq t \leq \tau_{0i} \end{cases} \quad (3-21)$$

where  $y$  is the produced quantity, and  $x_i$  is the maximum shortage accumulated for the  $i^{th}$  cycle.

And the positive inventory level, say  $I(t)$  is given by

$$I(t) = \begin{cases} 0 & , 0 \leq t \leq \tau_{1i} + \tau_{2i} \\ y - r [ t - (\tau_{1i} + \tau_{2i}) ] & , \tau_{1i} + \tau_{2i} \leq t \leq \tau_{1i} + \tau_{2i} + \tau_{3i} \\ Z_i - r [ t - (\tau_{1i} + \tau_{2i} + \tau_{3i}) ] & , \tau_{1i} + \tau_{2i} + \tau_{3i} \leq t \leq \tau_{0i} \end{cases} \quad (3-22)$$

where  $Z_i$  ( $Z_i = r\tau_{4i} = q_i - q_{1i} - r\tau_{3i}$ ) is the maximum stock accumulated during the production period.

### 3.5.1 Model formulation

Now, if  $\tau_{2i}$  is the cumulative time to produce a total quantity of  $y$  units, then letting  $y = q_{1i}$  in E.q (3-16), we obtain

$$\tau_{2i} = T_{11}mq_{1i} + (1-m)T_{1i} \frac{q_{1i}^{1-b}}{1-b} \quad (3-23)$$

Hence,

$$x_i = q_{1i} - r\tau_{2i} = q_{1i}(1 - rT_{11}m) - r(1-m)T_{1i} \frac{q_{1i}^{1-b}}{1-b} \quad (3-24)$$

Similarly, if  $\tau_{3i}$  is the cumulative time to produce a total quantity of  $y$  units, then

$$\begin{aligned}
t(y) &= \int_{q_{li}}^{q_{li}+y} [ T_{11}m + T_{li}(1-m)X^{-b} ] dX \\
&= T_{11}my + \frac{T_{li}(1-m)[ (q_{li} + y)^{1-b} - q_{li}^{1-b} ]}{1-b}
\end{aligned} \tag{3-25}$$

Letting  $y = q_i - q_{li}$  in E.q (3-25), we obtain

$$\tau_{3i} = T_{11}m(q_i - q_{li}) + (1-m)T_{li} \frac{(q_i^{1-b} - q_{li}^{1-b})}{1-b} \tag{3-26}$$

Therefore, the average shortage level with bounded learning (*ASBL*) for the  $i^{th}$  cycle is given by

$$\begin{aligned}
ASBL &= \int_0^{\tau_{0i}} S(t)dt = \int_0^{\tau_{2i}} [ x_i - y + r t ] dt + \frac{x_i^2}{2r} \\
&= \int_0^{\tau_{2i}} ( x_i + r t ) dt - \int_0^{q_i} y [ T_{11}m + T_{li}(1-m)y^{-b} ] dy + \frac{(q_{li} - r\tau_{2i})^2}{2r} \\
&= x_i\tau_{2i} - \left( \frac{T_{11}mq_{li}^2}{2} + \frac{T_{li}(1-m)q_{li}^{2-b}}{2-b} \right) + \frac{r\tau_{2i}^2}{2} + \frac{(q_{li} - r\tau_{2i})^2}{2r} \\
&= (q_{li} - r\tau_{2i})\tau_{2i} - \left( \frac{T_{11}mq_{li}^2}{2} + \frac{T_{li}(1-m)q_{li}^{2-b}}{2-b} \right) + r \frac{\tau_{2i}^2}{2} \\
&\quad + \frac{q_{li}^2}{2r} - q_{li}\tau_{2i} + r \frac{\tau_{2i}^2}{2} \\
&= q_{li}\tau_{2i} - r\tau_{2i}^2 - \left( \frac{T_{11}mq_{li}^2}{2} + \frac{T_{li}(1-m)q_{li}^{2-b}}{2-b} \right) + r\tau_{2i}^2 + \frac{q_{li}^2}{2r} - q_{li}\tau_{2i} \\
&= \frac{q_{li}^2}{2r} - \left( \frac{T_{11}mq_{li}^2}{2} + \frac{T_{li}(1-m)q_{li}^{2-b}}{2-b} \right) \\
&= \frac{q_{li}^2}{2r} [ 1 - rT_{11}m ] - \frac{T_{li}(1-m)q_{li}^{2-b}}{2-b}
\end{aligned} \tag{3-27}$$

Similarly, the average inventory level with bounded learning (*AIBL*) for the  $i^{th}$  cycle is given by

$$AIBL = \int_0^{\tau_{0i}} I(t)dt = \int_0^{\tau_{3i}} [y - r t] dt + \frac{Z_i^2}{2r} \quad (3-28)$$

Zhou and Lau [29] claimed that E.q (3-28) could be reduced after a bit of algebra, to

$$AIBL = \frac{(q_i - q_{1i})^2}{2r} \left( 1 - rT_{11}m \right) + \frac{T_{1i}(1-m)}{(1-b)} \left( (q_i - q_{1i})q_{1i}^{1-b} - \frac{q_i^{2-b}}{(2-b)} + \frac{q_{1i}^{2-b}}{(2-b)} \right) \quad (3-29)$$

However, this value is, in fact, incorrect. The correction is shown next, in details.

Below, we shall do some corrections to the above shortcoming given in Zhou and Lau [29].

### 3.6 Corrections of Zhou and Lau's [29] mathematical formulations <sup>a</sup>

Recalling E.q (3-29), the correct average inventory level with bounded learning (*AIBL*) for the  $i^{th}$  cycle is given by

$$\begin{aligned}
 AIBL &= \int_0^{\tau_{0i}} I(t)dt = \int_0^{\tau_{3i}} [y - rt]dt + \frac{Z_i^2}{2r} \\
 &= \int_0^{q_i - q_{1i}} y [T_{11}m + T_{1i}(1-m)(q_{1i} + y)^{-b}] dy - \int_0^{\tau_{3i}} rtdt + \frac{(q_i - q_{1i} - r\tau_{3i})^2}{2r} \\
 &= \int_0^{q_i - q_{1i}} y [T_{11}m] dy + \int_{q_{1i}}^{q_i} [w - q_{1i}] [T_{1i}(1-m)w^{-b}] dw \\
 &\quad - \int_0^{\tau_{3i}} rtdt + \frac{(q_i - q_{1i} - r\tau_{3i})^2}{2r}
 \end{aligned}$$

(where  $w = q_{1i} + y$ )

$$\begin{aligned}
 &= \int_0^{q_i - q_{1i}} y [T_{11}m] dy + \int_{q_{1i}}^{q_i} w [T_{1i}(1-m)w^{-b}] dw - \int_0^{\tau_{3i}} rtdt \\
 &\quad - \int_{q_{1i}}^{q_i} q_{1i} [T_{1i}(1-m)w^{-b}] dw + \frac{(q_i - q_{1i} - r\tau_{3i})^2}{2r} \\
 &= \frac{T_{11}m(q_i - q_{1i})^2}{2} + \frac{T_{1i}(1-m)(q_i^{2-b} - q_{1i}^{2-b})}{2-b} - r\frac{\tau_{3i}^2}{2} \\
 &\quad - \frac{q_{1i}T_{1i}(1-m)(q_i^{1-b} - q_{1i}^{1-b})}{1-b} + \frac{(q_i - q_{1i})^2}{2r} - (q_i - q_{1i})\tau_{3i} + r\frac{\tau_{3i}^2}{2} \\
 &= \frac{T_{11}m(q_i - q_{1i})^2}{2} + \frac{(q_i - q_{1i})^2}{2r} \\
 &\quad + \frac{T_{1i}(1-m)(q_i^{2-b} - q_{1i}^{2-b})}{2-b} - \frac{q_{1i}T_{1i}(1-m)(q_i^{1-b} - q_{1i}^{1-b})}{1-b} \\
 &\quad - (q_i - q_{1i}) \left[ T_{11}m(q_i - q_{1i}) + \frac{T_{1i}(1-m)(q_i^{1-b} - q_{1i}^{1-b})}{1-b} \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(q_i - q_{li})^2}{2r} \left( 1 - rT_{11}m \right) + \frac{T_{li}(1-m)(q_i^{2-b} - q_{li}^{2-b})}{2-b} \\
&\quad - \frac{q_i T_{li}(1-m)(q_i^{1-b} - q_{li}^{1-b})}{1-b} \\
&= \frac{(q_i - q_{li})^2}{2r} \left( 1 - rT_{11}m \right) + \frac{(1-b)T_{li}(1-m)q_i^{2-b} - (2-b)T_{li}(1-m)q_{li}^{2-b}}{(1-b)(2-b)} \\
&\quad - \frac{T_{li}(1-m)q_{li}^{2-b}}{(2-b)} + \frac{T_{li}(1-m)q_{li}^{1-b}q_i}{(1-b)} \\
&= \frac{(q_i - q_{li})^2}{2r} \left( 1 - rT_{11}m \right) - \frac{T_{li}(1-m)q_i^{2-b}}{(1-b)(2-b)} - \frac{T_{li}(1-m)q_{li}^{2-b}}{(2-b)} \\
&\quad + \frac{T_{li}(1-m)q_{li}^{1-b}q_i}{(1-b)} \tag{3-30}
\end{aligned}$$

E.q (3-30) does not agree with E.q (3-29), even though both equations are correct in the case where shortages are not allowed, i.e., if  $q_{li} = 0$ .

Additionally, Zhou and Lau's [29] dropped the following two constraints:

$$\int_0^{\tau_{2i}} [y - r t] dt = r\tau_{li} \tag{3-31}$$

$$\int_0^{\tau_{3i}} [y - r t] dt = r\tau_{4i} \tag{3-32}$$

Relation (3-31) insures the fact that the inventory level must have an equal value for time  $t = \tau_{li}$ , while relation (3-32) insures this fact for time  $t = \tau_{3i}$ .

<sup>a</sup> , This correction is due to the author.

### 3.7 Conclusion

In this chapter, we first introduced the general case in which bounded learning is considered. Then, we reviewed some recent studies concerning the effect of bounded learning on the optimal production quantity for single item. A viewpoint as well as corrections of the shortcomings appeared in these models are also given. In the next chapter, we shall incorporate the learning curve presented by De Jong [18] into the general production lot-sizing model introduced by Balkhi [4]. Further, this generalization will imply the correction of the model presented by Zhou and Lau [29], which we have shown above.

## Chapter 4

The effect of bounded learning on the optimal production lot size for deteriorating and partially exponential backordered items with time varying demand and deterioration rates

As indicated in the introduction of the previous chapter, the Wright's learning curve has some shortcomings, which can be overcome by the De Jong's bounded learning curve. In this chapter, we shall generalize the work of Balkhi [4] where we shall introduce a general learning production lot-sizing inventory model with the following features:

1. The learning curve to be used is that of De Jong [18].
2. A single item is produced in batches at an increasing rate, denoted by  $P(t)$ .
3. The items are subject to deterioration when they are effectively in stock and there is no repair or replacement of deteriorated items.
4. The demand and deterioration rates are known functions of time denoted by  $D(t)$  and  $\delta(t)$  respectively.
5. Shortages are allowed, but only a fraction  $\alpha(t) = e^{-t}$  is backordered and the rest  $(1 - \alpha(t))$  is lost,  $(0 < \alpha(t) \leq 1)$ , where  $t$  is the time up to backordering. This can be justified by the fact that, as the waiting time decreases, more and more customers are willing to get their orders as soon as the backlogged

demand reaches the system at the next production run, and vice versa (see also Papachristos and Skouri [21]).

6. We assume that the production is subject to a full transmission of learning and the production rate  $P(t)$  is defined in the natural sense as

$$P(t) = \frac{\text{Number of units produced up to time } t}{t} \quad (4 - 1)$$

and the initial production rate is assumed to be  $\frac{1}{t_1}$  ( $t_1$  is the time required to produce the first unit).

7. The cost parameters are as follows:

$c$  = Unit production cost, which includes materials, labors and manufacturing costs.

$h$  = Unit holding cost per unit per unit time.

$b$  = Unit shortages cost per unit per unit time for backordered items.

$s$  = Unit shortages cost per unit per unit time for lost items.

$k$  = Set up cost per set up.

For cycle  $j$ , we denote by  $I_j(t)$  the inventory level at time  $t$ .

In each cycle  $j$  ( $j = 1, 2, \dots$ ), the system starts operating at time  $T_{0j}$ , in which shortages are first allowed to occur at a demand rate  $\alpha(t) D(t)$  up to time  $T_{1j}$ , at which time a maximum backordered level of  $X_j$  units is reached. Then the

system starts production and the inventory level increases at a rate  $P(t) - D(t)$  in order to fulfill the demand and to clear the entire shortages of  $X_j$  units, where the inventory level becomes zero by time  $T_{2j}$ . Now the inventory level starts to go up with a rate  $P(t) - D(t) - \delta(t)I_j(t)$  until time  $T_{3j}$  where the inventory level reaches its maximum. Then the inventory level declines continuously at a rate  $D(t) - \delta(t)I_j(t)$  and becomes zero at time  $T_{4j}$  (the end of the cycle). The process is repeated. Figure 4 -1 depicts the inventory behavior during any cycle.

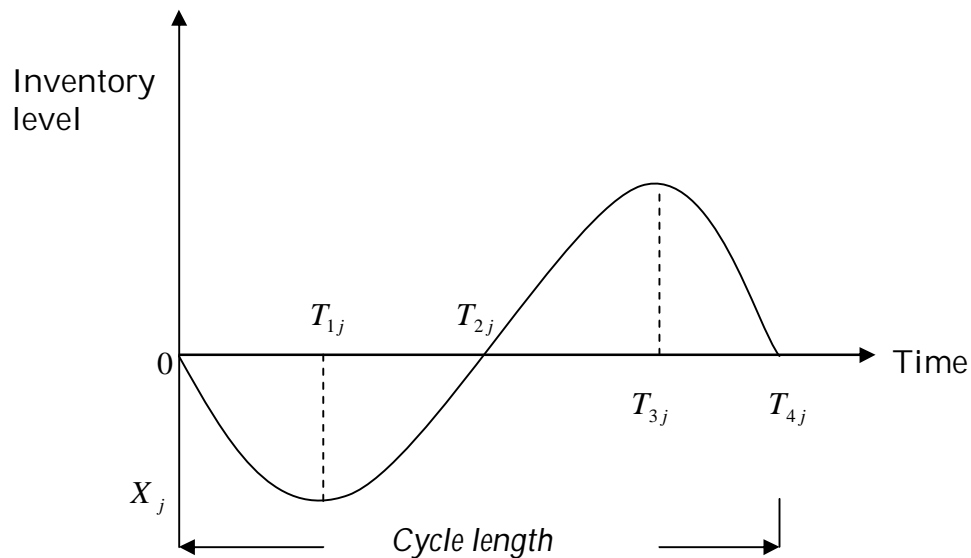


Figure 4 -1. Inventory variation of a general (EPQ) model in one cycle for bounded learning case and shortages

#### 4.1 Model formulation

The changes in the inventory level depicted in Figure 4 -1, are represented by the following differential equations

$$\frac{dI_j(t)}{dt} = -\alpha(t)D(t) \quad T_{0j} \leq t < T_{1j} \quad (4 -2)$$

with the initial condition  $I_j(T_{0j}) = 0$ ,

$$\frac{dI_j(t)}{dt} = -[P(t) - D(t)] \quad T_{1j} \leq t < T_{2j} \quad (4 -3)$$

with the ending condition  $I_j(T_{2j}) = 0$ ,

$$\frac{dI_j(t)}{dt} = P(t) - D(t) - \delta(t)I_j(t) \quad T_{2j} \leq t < T_{3j} \quad (4 -4)$$

with the initial condition  $I_j(T_{2j}) = 0$ , and

$$\frac{dI_j(t)}{dt} = -D(t) - \delta(t)I_j(t) \quad T_{3j} \leq t \leq T_{4j} \quad (4 -5)$$

with the ending condition  $I_j(T_{4j}) = 0$ .

Remark (1): The following first order differentiable equation,

$$y' + g(x)y = p(x)$$

$$\text{has the solution, } y = e^{-\int g(x)dx} \int e^{\int g(x)dx} p(x)dx + k e^{-\int g(x)dx} \quad (4 -6)$$

where  $k$  is an arbitrary constant.

By Remark (1), the solution of (4 -2) through (4 -5) can be found as follows:

Substituting (4 -2) into (4 -6) with respect to  $I_j(t)$  yields

$$I_j(t) = - \int_{T_{0j}}^t \alpha(t) D(t) dt + k \quad (4 -7)$$

with the boundary condition  $I_j(T_{0j}) = 0 \Rightarrow I_j(T_{0j}) = - \int_{T_{0j}}^{T_{0j}} \alpha(t) D(t) dt + k = 0$

$$\Rightarrow - \int_{T_{0j}}^{T_{0j}} \alpha(t) D(t) dt = 0 \Rightarrow k = 0$$

Let  $\alpha(t)D(t) = R(t)$  then equation (4 -7) yields

$$I_j(t) = - \int_{T_{0j}}^t R(u) du \quad T_{0j} \leq t < T_{1j} \quad (4 -8)$$

Taking into account, the boundary conditions of (4 -3), (4 -4), and (4 -5) then similar arguments yield the following solutions to (4 -3), (4 -4), and (4 -5)

$$I_j(t) = - \int_t^{T_{2j}} [P(u) - D(u)] du \quad T_{1j} \leq t < T_{2j} \quad (4 -9)$$

$$I_j(t) = e^{-g(t)} \int_{T_{2j}}^t [P(u) - D(u)] e^{g(u)} du \quad T_{2j} \leq t < T_{3j} \quad (4 -10)$$

$$I_j(t) = e^{-g(t)} \int_t^{T_{4j}} D(u) e^{g(u)} du \quad T_{3j} \leq t \leq T_{4j} \quad (4 -11)$$

respectively, where

$$g(t) = \int \delta(t) dt \quad (4 -12)$$

Let  $I(t_1, t_2) = \int_{t_1}^{t_2} I(u) du$ , then from (4 -10) and (4 -11) we have

$$I_j(T_{2j}, T_{3j}) = \int_{T_{2j}}^{T_{3j}} e^{-g(t)} \left( \int_{T_{2j}}^t [P(u) - D(u)] e^{g(u)} du \right) dt \quad (4 -13)$$

$$I_j(T_{3j}, T_{4j}) = \int_{T_{3j}}^{T_{4j}} e^{-g(t)} \left( \int_t^{T_{4j}} D(u) e^{g(u)} du \right) dt \quad (4 -14)$$

Now using integration by parts, E.q (4 -13) can be solved as follows

Let  $e^{-g(t)} dt = dG$  then,

$$G(t) = \int e^{-g(t)} dt \quad (4 -15)$$

$$\int_{T_{2j}}^t [P(u) - D(u)] e^{g(u)} du = v \Rightarrow dv = [P(t) - D(t)] e^{g(t)} dt$$

$$\int_{T_{2j}}^{T_{3j}} v dG = [vG] \Big|_{T_{2j}}^{T_{3j}} - \int_{T_{2j}}^{T_{3j}} G dv$$

$$= \left( \int_{T_{2j}}^t [P(u) - D(u)] e^{g(u)} du \right) (G(t)) \Big|_{T_{2j}}^{T_{3j}} - \int_{T_{2j}}^{T_{3j}} (G(t)) [P(t) - D(t)] e^{g(t)} dt$$

$$= \left( \int_{T_{2j}}^{T_{3j}} [P(u) - D(u)] e^{g(u)} du \right) (G(T_{3j})) - 0 - \left( \int_{T_{2j}}^{T_{3j}} [P(u) - D(u)] e^{g(u)} du \right) (G(u))$$

$$= \left( \int_{T_{2j}}^{T_{3j}} [G(T_{3j} - G(u)) [P(u) - D(u)] e^{g(u)} du \right)$$

It follows that

$$I_j(T_{2j}, T_{3j}) = \int_{T_{2j}}^{T_{3j}} [G(T_{3j}) - G(u)] [P(u) - D(u)] e^{g(u)} du \quad (4 -16)$$

By similar way we find

$$I_j(T_{3j}, T_{4j}) = \int_{T_{3j}}^{T_{4j}} [G(u) - G(T_{3j})] D(u) e^{g(u)} du \quad (4 -17)$$

$$I_j(T_{0j}, T_{1j}) = \int_{T_{0j}}^{T_{1j}} (T_{1j} - u) R(u) du \quad (4 -18)$$

$$I_j(T_{1j}, T_{2j}) = \int_{T_{1j}}^{T_{2j}} [u - T_{1j}] [P(u) - D(u)] du \quad (4 -19)$$

Note that we can set  $T_{0j} = 0$  without loss of generality. Thus, the cost components for the given inventory system are as follows:

Items cost  $= c \int_{T_{1j}}^{T_{3j}} P(u) du$ . Note that this cost includes the deteriorated and the consumed items.

Holding cost  $= h [ I_j(T_{2j}, T_{3j}) + I_j(T_{3j}, T_{4j}) ]$ .

Shortage cost for backordered items  $= b [ I_j(0, T_{1j}) + I_j(T_{1j}, T_{2j}) ]$ .

Shortage cost for lost items  $= s(1 - \alpha(u)) \int_0^{T_{1j}} D(u) du$ .

Set up cost per set up  $= k$ .

Therefore the total cost per unit time of the underlying inventory system during the cycle  $[0, T_{4j}]$ , which consists of the production cost, inventory holding cost, shortages costs for the backordered and the lost items and set up cost, as a function of  $T_{1j}, T_{2j}, T_{3j}$  and  $T_{4j}$ , say  $L(T_{1j}, T_{2j}, T_{3j}, T_{4j})$ , is given by

$$\begin{aligned}
 L(T_{1j}, T_{2j}, T_{3j}, T_{4j}) = & \frac{1}{T_{4j}} \left\{ c \int_{T_{1j}}^{T_{3j}} P(u) du + h \left[ \int_{T_{2j}}^{T_{3j}} \{G(T_{3j}) - G(u)\} \{P(u) - D(u)\} \right. \right. \\
 & \times e^{g(u)} du + \left. \int_{T_{3j}}^{T_{4j}} \{G(u) - G(T_{3j})\} D(u) e^{g(u)} du \right] \\
 & + b \left[ \int_0^{T_{1j}} (T_{1j} - u) R(u) du + \int_{T_{1j}}^{T_{2j}} (u - T_{1j}) (P(u) - D(u)) du \right] \\
 & \left. + s \int_0^{T_{1j}} D(u) du - s \int_0^{T_{1j}} R(u) du + k \right\} \quad (4-20)
 \end{aligned}$$

where  $g(u)$  is given by (4 -12),  $G(u)$  is given by (4 -15),  $\alpha(u) = e^{-u}$ , and  $R(u) = \alpha(u)D(u)$ .

Our goal is to find  $T_{1j}, T_{2j}, T_{3j}$  and  $T_{4j}$ , that minimize  $L(T_{1j}, T_{2j}, T_{3j}, T_{4j})$ , where  $L(T_{1j}, T_{2j}, T_{3j}, T_{4j})$  is given by (4 -20). However, the variables  $T_{1j}, T_{2j}, T_{3j}$  and  $T_{4j}$  are related to each other through the following relations

$$0 < T_{1j} < T_{2j} < T_{3j} < T_{4j} \quad (4 -21)$$

$$\int_0^{T_{1j}} R(u) du = \int_{T_{1j}}^{T_{2j}} [P(u) - D(u)] du \quad (4 -22)$$

$$e^{-g(T_{3j})} \int_{T_{2j}}^{T_{3j}} [P(u) - D(u)] e^{g(u)} du = e^{-g(T_{3j})} \int_{T_{3j}}^{T_{4j}} D(u) e^{g(u)} du \quad (4 -23)$$

Relation (4 -21) represents the natural monotonicity constraints, since otherwise the given problem would have no meaning. Relations (4 -22) and (4 -23) ensure the fact that the inventory levels must have equal values for  $t = T_{1j}$  and for  $t = T_{3j}$  respectively. Thus, our goal is to solve the following optimization problem, which we shall call problem (p)

$$(p) = \begin{cases} \text{Minimize } L(T_{1j}, T_{2j}, T_{3j}, T_{4j}) & \text{subject to (4-21)} \\ \text{and } h_1 = 0 \ \& \ h_2 = 0 \end{cases}$$

where  $L(T_{1j}, T_{2j}, T_{3j}, T_{4j})$  is given by (4 -20) and  $h_1$ ,  $h_2$  are respectively given by

$$h_1 = \int_0^{T_{1j}} R(u) du - \int_{T_{1j}}^{T_{2j}} [P(u) - D(u)] du$$

$$h_2 = \int_{T_{2j}}^{T_{3j}} [P(u) - D(u)] e^{g(u)} du - \int_{T_{3j}}^{T_{4j}} D(u) e^{g(u)} du$$

As mentioned in the previous chapter, the solution of similar problem has been introduced in details in Balkhi [2, 3, and 4]. The most important point to be reminded from Balkhi [2, 3, and 4] is that if we temporarily ignore the monotony constrains (4 -21) and call the resulting problem as  $(p_1)$  then (4 -21) do satisfy any solution of  $(p_1)$ . Hence  $(p)$  and  $(p_1)$  are equivalent. Next, we shall incorporate the bounded learning curve given by De Jong [18] in the above general (EPQ) model.

This learning curve is given as follows: (see De Jong [18])

$$t_{ij} = t_{11}m + (1-m)t_{1j} i^{-r} \quad (4 -24)$$

where  $t_{ij}$  is the time required to produce the  $i^{th}$  unit in cycle  $j$ ,  $t_{1j}$  is the variable component of the time needed to produce the first unit in the  $j^{th}$  cycle,  $t_{1j} = t_{11} (1 + \sum_{n=1}^{j-1} q_n)^{-r}$  where  $q_n$  is the quantity produced in production cycle  $n$ ,  $m$  is the incompressibility factor ( $0 \leq m \leq 1$ ); and  $r$  is the learning slope reflecting the decrease in the production time required per unit. (Recall the justification of using this learning curve with the above parameters in the previous chapter).

#### 4.2 Another (approximated) form for E.q (4 -24)

Let  $t_j$  be the time required to produce the  $i$  units in the  $j^{th}$  cycle then

$$t_j = \sum_{k=1}^i \left( t_{11} m + (1-m) t_{1j} k^{-r} \right) \quad (4 -25)$$

$$t_j = \sum_{k=1}^i t_{11} m + \sum_{k=1}^i \left( (1-m) t_{1j} k^{-r} \right) \quad (4 -26)$$

$$t_j \approx t_{11} m i + (1-m)t_{1j} \int_0^i k^{-r} dk \quad (4 -27)$$

$$t_j = t_{11} m i + (1-m) t_{1j} \frac{i^{1-r}}{1-r} \quad (4 -28)$$

Recalling (4 -1), then by definition of  $P(t)$  it follows that

$$P(t_j) = \frac{i}{t_j} = \frac{1-r}{t_{11} m (1-r) + t_{1j} (1-m) i^{-r}} \quad (4 -29)$$

Now, let  $Q_j$  be the amount of units to be produced in the interval  $[T_{1j}, T_{3j}]$

then, from (4 -28) we have

$$T_{3j} - T_{1j} = t_{11} m Q_j + (1-m)t_{1j} \frac{Q_j^{1-r}}{1-r} \quad (4 -30)$$

In addition, from the definition of  $X_j$  we have

$$X_j = \int_0^{T_{1j}} R(u) du \quad (4 -31)$$

Note that the right hand side (RHS) of (4 -31) is an increasing function of  $T_{1j}$ ,

so  $T_{1j}$  can be uniquely determined as a function of  $X_j$ , say

$$T_{1j} = f_{1j}(X_j) \quad (4 -32)$$

From (4 -32) and (4 -22),  $T_{2j}$  can be uniquely determined as a function of  $T_{1j}$ ,

hence of  $X_j$ , say

$$T_{2j} = f_{2j}(X_j) \quad (4 -33)$$

Substituting (4 -32) in (4 -30) we find that  $T_{3j}$  can be uniquely determined as a

function of  $X_j$  &  $Q_j$ , say

$$T_{3j} = f_{3j}(X_j, Q_j) \quad (4 -34)$$

From (4 -33), (4 -34), and (4 -23),  $T_{4j}$  can be uniquely determined as a function of  $T_{3j}$  hence of  $X_j$  &  $Q_j$ , say

$$T_{4j} = f_{4j}(X_j, Q_j) \quad (4 -35)$$

Substituting (4 -22), (4 -23), and (4 -32) through (4 -35) in (4 -20) we obtain the following unconstrained problem with the variables  $X_j$  &  $Q_j$

$$\begin{aligned} \text{minimize } W(X_j, Q_j) = & \frac{1}{f_{4j}} \left\{ c \int_{f_{1j}}^{f_{3j}} P(u) du \right. \\ & + h \left[ \int_{f_{2j}}^{f_{3j}} -G(u)\{P(u)-D(u)\}e^{g(u)} du + \int_{f_{3j}}^{f_{4j}} G(u)D(u) e^{g(u)} du \right] \\ & + h G(f_{3j}) \left[ \int_{f_{2j}}^{f_{3j}} \{P(u)-D(u)\}e^{g(u)} du - \int_{f_{3j}}^{f_{4j}} D(u) e^{g(u)} du \right] \\ & + b \left[ \int_0^{f_{1j}} -u R(u) du + \int_{f_{1j}}^{f_{2j}} u (P(u)-D(u)) du \right] \\ & + b f_{1j} \left[ \int_0^{f_{1j}} R(u) du - \int_{f_{1j}}^{f_{2j}} (P(u)-D(u)) du \right] \\ & \left. + s \int_0^{f_{1j}} D(u) du - s \int_0^{f_{1j}} R(u) du + k \right\} \quad (4-36) \end{aligned}$$

The unconstrained case happens since (4 -32) through (4 -35) were obtained by direct substitution one in another.

Recalling (4 -22) & (4 -23) then (4 -36) reduces to

$$\begin{aligned} \text{minimize } W(X_j, Q_j) = & \frac{1}{f_4} \left\{ c \int_{f_1}^{f_3} P(u) du \right. \\ & + h \left[ \int_{f_2}^{f_3} -G(u)\{P(u)-D(u)\}e^{g(u)} du + \int_{f_3}^{f_4} G(u)D(u) e^{g(u)} du \right] \\ & + b \left[ \int_0^{f_1} -u R(u) du + \int_{f_1}^{f_2} u (P(u)-D(u)) du \right] \\ & \left. + s \int_0^{f_1} D(u) du - s \int_0^{f_1} R(u) du + k \right\} \quad (4-37) \end{aligned}$$

Then, the necessary conditions for having optima are

$$\frac{\partial W}{\partial X_j} = 0 \quad \text{and} \quad \frac{\partial W}{\partial Q_j} = 0 \quad (4 - 38)$$

Now, let  $W = \frac{w}{f_4}$  then

$$\frac{\partial W}{\partial X_j} = \frac{w'_{X_j} f_4 - f'_{4,X_j} w}{f_4^2}$$

Also,

$$\frac{\partial W}{\partial Q_j} = \frac{w'_{Q_j} f_4 - f'_{4,Q_j} w}{f_4^2}$$

Note that, (4 - 38) are equivalent to

$$w'_{X_j} f_4 - f'_{4,X_j} w = 0 \quad (4 - 39)$$

$$w'_{Q_j} f_4 - f'_{4,Q_j} w = 0 \quad (4 - 40)$$

respectively, where  $w'_Y$  and  $f'_{i,Y}$  are the derivatives of  $w$  and  $f_i$  with respect to (w.r.t)  $Y$ .

Now, multiplying (4 - 39) by  $-f'_{4,Q_j}$  and (4 - 40) by  $f'_{4,X_j}$  and summing we have

$$\begin{aligned} -f'_{4,Q_j} w'_{X_j} f_4 + f'_{4,Q_j} f'_{4,X_j} w &= 0 \\ f'_{4,X_j} w'_{Q_j} f_4 - f'_{4,X_j} f'_{4,Q_j} w &= 0 \\ \Rightarrow f'_{4,X_j} w'_{Q_j} f_4 &= f'_{4,Q_j} w'_{X_j} f_4 \\ \Rightarrow f'_{4,X_j} w'_{Q_j} &= f'_{4,Q_j} w'_{X_j} \end{aligned} \quad (4 - 41)$$

Taking the derivative of both sides of (4 - 23), (w.r.t)  $Q_j$  we have

$$\begin{aligned} f'_{3,Q_j} P(f_3) e^{g(f_3)} - f'_{3,Q_j} D(f_3) e^{g(f_3)} - f'_{2,Q_j} P(f_2) e^{g(f_2)} + f'_{2,Q_j} D(f_2) e^{g(f_2)} = \\ f'_{4,Q_j} D(f_4) e^{g(f_4)} - f'_{3,Q_j} D(f_3) e^{g(f_3)} \end{aligned}$$

Since  $T_{2j}$  is not dependent on  $Q_j$ , so  $f'_{2,Q_j} = 0$ . Thus, we have

$$f'_{3,Q_j} P(f_3) e^{g(f_3)} = f'_{4,Q_j} D(f_4) e^{g(f_4)} \quad (4 - 42)$$

From (4 -32) through (4 -36) we obtain

$$\begin{aligned} w'_{Q_j} = & cf'_{3,Q_j} P(f_3) - hf'_{3,Q_j} G(f_3) [P(f_3) - D(f_3)] e^{g(f_3)} \\ & + hf'_{4,Q_j} G(f_4) D(f_4) e^{g(f_4)} - hf'_{3,Q_j} G(f_3) D(f_3) e^{g(f_3)} \end{aligned}$$

which, from (4 - 42) reduces to

$$w'_{Q_j} = f'_{3,Q_j} P(f_3) \left( c + h e^{g(f_3)} [G(f_4) - G(f_3)] \right) \quad (4 - 43)$$

Also, from (4 -32) through (4 -36) we obtain

$$\begin{aligned} w'_{X_j} = & c [ f'_{3,X_j} P(f_3) - f'_{1,X_j} P(f_1) ] \\ & + h \left( [ -f'_{3,X_j} G(f_3) ] [ P(f_3) - D(f_3) ] e^{g(f_3)} \right. \\ & + [ f'_{2,X_j} G(f_2) ] [ P(f_2) - D(f_2) ] e^{g(f_2)} \\ & \left. + f'_{4,X_j} G(f_4) D(f_4) e^{g(f_4)} - f'_{3,X_j} G(f_3) D(f_3) e^{g(f_3)} \right) \\ & + b \{ -f'_{1,X_j} f_1 R(f_1) + f'_{2,X_j} f_2 [ P(f_2) ] - D(f_2) - f'_{1,X_j} f_1 [ P(f_1) ] - D(f_1) \} \\ & + sf'_{1,X_j} D(f_1) - sf'_{1,X_j} R(f_1) \end{aligned} \quad (4-44)$$

From (4 -22), (4 -32), (4 -33), and (4 -23) we respectively have

$$\begin{aligned} f'_{1,X_j} R(f_1) &= f'_{2,X_j} [P(f_2) - D(f_2)] - f'_{1,X_j} [P(f_1) - D(f_1)] \\ \Rightarrow f'_{1,X_j} f_1 R(f_1) &= f'_{2,X_j} f_1 [P(f_2) - D(f_2)] - f'_{1,X_j} f_1 [P(f_1) - D(f_1)] \end{aligned} \quad (4 - 45)$$

$$\begin{aligned} f'_{3,X_j} P(f_3) e^{g(f_3)} - f'_{2,X_j} [P(f_2) - D(f_2)] e^{g(f_2)} &= f'_{4,X_j} D(f_4) e^{g(f_4)} \\ \Rightarrow f'_{3,X_j} P(f_3) e^{g(f_3)} G(f_4) - f'_{2,X_j} [P(f_2) - D(f_2)] e^{g(f_2)} G(f_4) &= \\ f'_{4,X_j} D(f_4) e^{g(f_4)} G(f_4) & \end{aligned} \quad (4-46)$$

Substituting (4 - 45) & (4 - 46) in  $w'_{X_j}$  we obtain

$$\begin{aligned} w'_{X_j} = & c [ f'_{3,X_j} P(f_3) - f'_{1,X_j} P(f_1) ] + hf'_{3,X_j} P(f_3) e^{g(f_3)} [ G(f_4) ] - G(f_3) ] \\ & + f'_{2,X_j} [ P(f_2) - D(f_2) ] \{ b( f_2 - f_1 ) - he^{g(f_2)} [ G(f_4) ] - G(f_2) \} \\ & + sf'_{1,X_j} D(f_1) - sf'_{1,X_j} R(f_1) \end{aligned} \quad (4-47)$$

From (4 -30)  $f'_{3,X_j} = f'_{1,X_j}$ , then (4 - 47) reduces to

$$\begin{aligned} w'_{X_j} = & c [ f'_{1,X_j} P(f_3) - f'_{1,X_j} P(f_1) ] + hf'_{1,X_j} P(f_3) e^{g(f_3)} [ G(f_4) ] - G(f_3) ] \\ & + f'_{2,X_j} [ P(f_2) - D(f_2) ] \{ b( f_2 - f_1 ) - he^{g(f_2)} [ G(f_4) ] - G(f_2) \} \\ & + sf'_{1,X_j} D(f_1) - sf'_{1,X_j} R(f_1) \end{aligned} \quad (4-48)$$

From (4 - 41), (4 - 42), (4 - 43), and (4 - 47) we have

$$\begin{aligned} f'_{4,X_j} f'_{3,Q_j} P(f_3) \left( c + he^{g(f_3)} [ G(f_4) - G(f_3) ] \right) = \\ f'_{4,Q_j} c [ f'_{1,X_j} P(f_3) - f'_{1,X_j} P(f_1) ] + hf'_{1,X_j} P(f_3) e^{g(f_3)} [ G(f_4) ] - G(f_3) ] \\ + f'_{2,X_j} [ P(f_2) - D(f_2) ] \{ b( f_2 - f_1 ) - he^{g(f_2)} [ G(f_4) ] - G(f_2) \} \\ + sf'_{1,X_j} D(f_1) - sf'_{1,X_j} R(f_1) \end{aligned} \quad (4-49)$$

$$\text{Also (4 -40) } \Leftrightarrow w = \frac{w'_{Q_j} f_4}{f'_{4,Q_j}}$$

$$\text{Or } W = \frac{w}{f_4} = \frac{w'_{Q_j}}{f'_{4,Q_j}} \quad (4-50)$$

where  $W$  is given by (4 -36) and  $w'_{Q_j}$  is given by (4 - 42). Recalling (4 -31) and

the definition of  $P(t)$  as introduced in (4 -1), we have

$$P(f_1) = \frac{1}{t_{1j}} \quad (4-51)$$

$$P(f_2) = \frac{X_j + \int_{f_1}^{f_2} R(t) dt}{f_2} = \frac{X_j + \int_0^{f_2} R(t) dt - \int_0^{f_1} R(t) dt}{f_2}$$

$$\Rightarrow P(f_2) = \frac{\int_0^{f_2} R(t)dt}{f_2} \quad (4 - 52)$$

$$P(f_3) = \frac{Q_j}{f_3} \quad (4 - 53)$$

The two equations (4 - 49) & (4 - 50) can, now be used to determine the optimal values of  $X_j$  &  $Q_j$ . Then the minimum total cost can be determined from (4 - 50).

### 4.3 Conclusion

In this chapter, we have incorporated the learning curve presented by De Jong [18] into the work of Balkhi [4], where we presented a general production lot-sizing inventory model for deteriorating items with time varying demand and deterioration rates, shortages are allowed and partially backordered with exponential rate. Both demand and deterioration rates are general functions of time. The production rate is defined as the number of units produced per unit time. A closed form for the total relevant costs is derived, and rigorous mathematical methods that guide to a minimum total cost of the underlying inventory system are introduced. This seems to be the first time where such general model is investigated and derived. As mentioned in the introduction of the thesis that the loss of performance over the production break has its influence on the cost and /or the time required to produce the unit due to the forgetting phenomenon. Such phenomenon requires a different curve from those introduced so far. The effect of the forgetting phenomenon will be introduced in the next chapter.

## Chapter 5

### Some inventory models under learning and forgetting effects

#### 5.1 Introduction

As shown in chapter 2, the learning phenomenon implies that the performance of a system engaged in repetitive manufacturing operations improves with time. Recalling that Wright's learning curve is given by

$$t_n = t_1 n^{-r} \quad (5-1)$$

where  $t_n$  is the time required to produce the  $n^{\text{th}}$  unit,  $t_1$  is the time required to produce the first unit,  $n$  is the production count, and  $r$  is the slope of the learning curve. The learning curve introduced by Wright [27] is a common used model, because it has a simple and applicable mathematical form. However, it implies a decrease in the cumulative time per unit as the quantity produced approaches to  $n$ . On the other hand, a break in production will have an adverse effect on  $t_n$ . For example, it is reasonable to assume that if a large amount of time has elapsed between subsequent production runs, then, we would not follow the same learning curve at the point where production resumes. The production rate at the re-commencement of production might not be as high as when the production ceased, and hence, the cost and / or the time needed to produce the first unit in the next production run will increase as the length of the break increases. This loss of performance over the

production break is due to the forgetting phenomenon. Carlson and Rowe [6] described the forgetting or interruption portion of the learning cycle by a negative decay function equivalent to the decay observed in electrical losses in condensers. The forgetting curve relation presented by Carlson and Rowe [6], is expressed as follows:

$$\hat{t}_{\mu} = \hat{t}_1 \mu^l \quad (5-2)$$

where  $\hat{t}_{\mu}$  is the equivalent time for the  $\mu^{th}$  unit of lost experience at which performance has been lost over the production break, the quantity  $\mu$  is the amount of units that would have been produced assuming a continuous production run,  $\hat{t}_1$  is the equivalent variable component of the time needed for the first unit of the forgetting curve, and  $l$  is the slope of the forgetting curve reflecting the increase in the production time required per unit.

Next, we shall review some recent studies concerning the effect of learning and forgetting on the manufacturing lot-size problem.

Jaber and Bonney [12] extended the work of Salameh et al [22] by developing a mathematical model that describes the learning-forgetting relationship, referred to as the learning-forgetting curve model (LFCM).

## 5.2 The learn forget curve model

In order to establish the (LFC), Jaber and Bonney [12] assumed that  $q$  units are to be manufactured in each inventory cycle. Interruption occurs immediately after producing the  $q^{\text{th}}$  unit. They also assumed that the forgetting slope is dependent on three factors. These factors are the equivalent accumulated output of continuous production at the point of interruption, the minimum break to which the manufacturer assumes total forgetting, and the learning slope. The (LFC) is depicted in Figure 5-1.

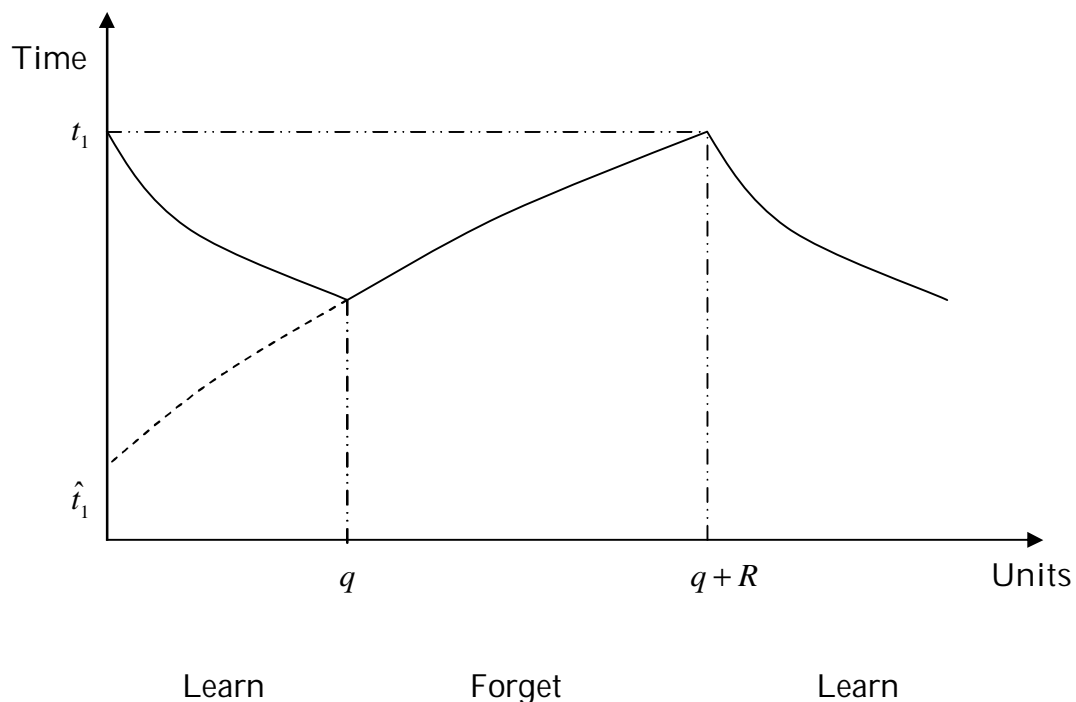


Figure 5-1. The effect of learning and forgetting on the time required to produce a unit

Jaber and Bonney [12] stated that the production on the day in which production stops in the first cycle is equal to the production on the same day of the forgetting cycle, that is

$$t_1 q^{-r} = \hat{t}_1 q^l$$

Solving for  $\hat{t}_1$  gives

$$\hat{t}_1 = t_1 q^{-(r+l)} \quad (5-3)$$

Substituting E.q (5-3) into E.q (5-2), the time required to produce the  $\mu^{\text{th}}$  unit after experiencing  $q$  units is given by

$$\hat{t}_\mu = t_1 q^{-(r+l)} \mu^l \quad (5-4)$$

They assumed that total forgetting occurs at a point  $\mu = q + R$ , then E.q (5-4) is set equal to  $t_1$ , or

$$t_1 = t_1 q^{-(r+l)} (q + R)^l$$

Solving for  $l$  gives

$$l = \frac{r \log(q)}{\log(q + R) - \log(q)} \quad (5-5)$$

The production break time  $t_B$ , which is the elapsed time between producing  $q$  units and  $q + R$  units is given by

$$t_B = \int_q^{q+R} t_1 X^{-r} dX = \frac{t_1}{1-r} \left[ (q + R)^{1-r} - q^{1-r} \right] \quad (5-6)$$

Similarly, if  $t_p$  is the cumulative time to produce a total quantity of  $q$  units, then

$$t_p = \int_0^q t_1 X^{-r} dX = \frac{t_1}{1-r} q^{1-r} \quad (5-7)$$

Solving for  $q + R$  in E.q (5-6) yields

$$(q + R) = \left[ \frac{1-r}{t_1} t_B + q^{1-r} \right]^{\frac{1}{1-r}} \quad (5-8)$$

Substituting E.q (5-7) into E.q (5-8) gives

$$(q + R) = q [ C + 1 ]^{\frac{1}{1-r}} \quad (5-9)$$

where  $C = t_B / t_p$

$C$ , is the minimum value of the ratio of the break time to production time that will achieve total forgetting. If the system experiences smaller interruption periods  $t_b$ , where  $0 \leq t_b < t_B$ , then the time required to produce the first unit in the next production run is greater than the time required to produce the last unit in the previous cycle, but less than the time required to produce the first unit in the first cycle. Substituting E.q (5-5) into E.q (5-9) gives

$$l = \frac{r(1-r)\log(q)}{\log(C+1)} \quad (5-10)$$

In E.q (5-10), the value of the forgetting slope  $l$  is zero whenever the learning slope  $r$  is either zero or one. Jaber and Bonney [12] stated that these two extreme cases corresponds to, the case where there is no learning involved, then there is nothing to forget, and the case where the system improves rapidly (i.e. the learning slope is very large), where then the forgetting slope is negligible.

They defined  $\alpha$ , ( $0 \leq \alpha \leq q$ ) as the amount of equivalent units of experience at the beginning of the production run after an interruption period of length  $t_b$ .

The value of  $\alpha$  can be obtained by equating E.q (5-1) and E.q (5-2) to give

$$\alpha = q^{(r+1)/r} (q+s)^{-1/r} \quad (5-11)$$

where  $s \leq R$  when  $t_b \leq t_B$ .

Therefore, the time required to produce the first unit in the next production run is given by

$$t_{q+1} = \left[ q^{(r+1)/r} (q+s)^{(r+1)/r} + 1 \right]^{-r} \quad (5-12)$$

### 5.3 Numerical example

An inventory situation with the following parameters was considered in Jaber and Bonney [12].

Time to produce the first unit  $t_1 = 0.2 \text{ days}$

Total forgetting  $t_B = 300 \text{ days}$

Production break  $t_b = 10 \text{ days}$

Learning slope  $r = 0.152$

Interruption occurs immediately after producing  $q = 200$  units.

The cumulative time to produce a total quantity of  $q$  units is given by

$$t_p = t(q) = \int_0^q t_1 X^{-r} dX = \frac{0.2}{1-0.152} 200^{1-0.152} = 21.08 \text{ days}$$

Then  $C = t_B / t_p = 300 / 21.08 = 14.23$ .

Therefore, the forgetting slope is given by

$$l = \frac{0.152 \times (1 - 0.152) \times \log(200)}{\log(14.23 + 1)} = 0.251$$

The total amount of output ( $q + s$ ) that would have been accumulated if the process had not been interrupted for a period of  $t_b = 10$  days can be determined from E.q (5-8) as

$$(q + s) = \left[ \frac{1 - 0.152}{0.2} \times 10 + 200^{1 - 0.152} \right]^{\frac{1}{1 - 0.152}} = 316 \text{ units}$$

This means that the system has lost the opportunity of producing 116 additional units over the interrupted period of  $t_b = 10$  days .

The amount  $\alpha$  , of equivalent units of experience at the beginning of the next production run after an interruption period of length  $t_b$  is equal to

$$\alpha_2 = q^{(r+l)/r} (q + s)^{-l/r} = 200^{(0.251+0.152)/0.152} \times (316)^{-0.251/0.152} = 94 \text{ units}$$

Therefore, the time required to produce the first unit in the next production run is given by

$$t_{201} = 2 \times (94 + 1)^{-0.152} = .1001 \text{ days}$$

The process is repeated.

#### 5.4 The optimal lot size problem

Jaber and Bonney [12] extended the work of Salameh et al [22] by incorporated the (LFCM) to solve the example presented by Salameh et al [22] as follows:

Recalling that the total cost per unit time  $TCU$  as given by Salameh et al [22] is equal to

$$TCU(q_i) = \frac{\gamma t_1 q_i^{-b} r}{1-b} + d_m r + h \left[ \frac{q}{2} - \frac{t_1 q_i^{1-b} r}{(1-b)(2-b)} \right] + \frac{kr}{q_i}$$

The Newton-Raphson method was used to solve the above nonlinear, algebraic total cost. Table 5-1, gives the computations of the optimum values for the time required to produce the first unit, the optimal production quantities and the corresponding time, and the maximum stock accumulated for nine successive cycles.

Table 5-1. Optimum quantities under partial transmission of learning

Cycle $i$	$t_{1i}$	$q_i^*$	$t(q_i^*)$	$Z_i$
1	0.0625	216	8.75	111
2	0.0406	188	5.03	128
3	0.0395	187	4.86	129
4	0.0395	186	4.80	129
5	0.0391	186	4.77	129
6	0.0389	186	4.76	129
7	0.0389	186	4.76	129
8	0.0388	186	4.76	129
9	0.0388	186	4.76	129

The (LFCM) was tested and shown to be consistent with the model presented by Globerson et al [9] with less than 1% deviation (see also Globerson et al [9]). With the (LFCM), it is possible to determine the value of the forgetting rate once its mathematical form is assumed. They showed that forgetting had an adverse effect because of the drop in labor productivity.

Remark

It is to be noted here that Jaber and Bonney [12] dropped the following constraint

$$\int_0^{\tau_1} \left[ \left( \frac{1-b}{T_1} t \right)^{\frac{1}{1-b}} - r t \right] dt = r \tau_2$$

which insures the fact that the inventory level must have an equal value for time  $t = \tau_1$ .

Jaber and Bonney [15] developed three models for the infinite and finite planning horizon. The infinite planning horizon model is repeating the example solved by Jaber and Bonney [12] but using different parameters. The two models developed for the infinite planning horizon were for equal and unequal lot size cases. They showed that under partial transmission of learning the optimal policy was to carry fewer inventories in later lots.

Jaber and Bonney [14] studied the effect of learning and forgetting on the optimal manufactured quantity with the consideration of intracycle, within cycle, backorders. They showed that the presence of interruptions results in longer cycle runs causing further increase on labor and inventory costs. These costs tend to be more critical in the case when only a partial transmission of learning is assumed.

Next, we shall introduce a general production lot-sizing inventory model under the effects of learning and forgetting.

The effects of learning and forgetting on the optimal production lot size for deteriorating items with time varying demand and deterioration rates

We shall generalize the work of Jaber and Bonney [12] under the following assumptions:

1. A single item is produced in batches at an increasing rate, denoted by  $P(t)$ .
2. The items are subject to deterioration and there is no repair or replacement of deteriorated items.
3. The demand and deterioration rates are known functions of time denoted by  $D(t)$  and  $\delta(t)$  respectively.
4. We assume that the production is subject to learn-forget-learn relationship and the production rate  $P(t)$  is defined in the more natural sense as

$$P(t) = \frac{\text{Number of units produced up to time } t}{t} \quad (5-13)$$

In addition, the initial production rate is assumed to be  $1/t_1$ , where  $t_1$  is the time required to produce the first unit.

5. The cost parameters are as follows:

$c$  = Unit production cost, which includes materials, labors, and manufacturing costs.

$h$  = Unit holding cost per unit per unit time.

$k$  = Set up cost per set up.

For cycle  $j$ ,  $I_j(t)$  denotes the inventory level at time  $t$ . In each cycle  $j$  ( $j=1,2,\dots$ ), the system starts operating at time  $T_{0j}$ , in which production takes place and the inventory level increases at a rate  $P(t) - D(t) - \delta(t)I_j(t)$  until time  $T_{1j}$  where the inventory level reaches its maximum. Now, the production is stopped and the system is subject to forgetting from time  $T_{1j}$  up to time  $T_{2j}$ . During this intermittent period, the inventory level declines continuously at a rate  $D(t) - \delta(t)I_j(t)$  and becomes zero at time  $T_{2j}$  (the end of the cycle). The process is repeated. The behavior of such system is depicted in figure 5-2.

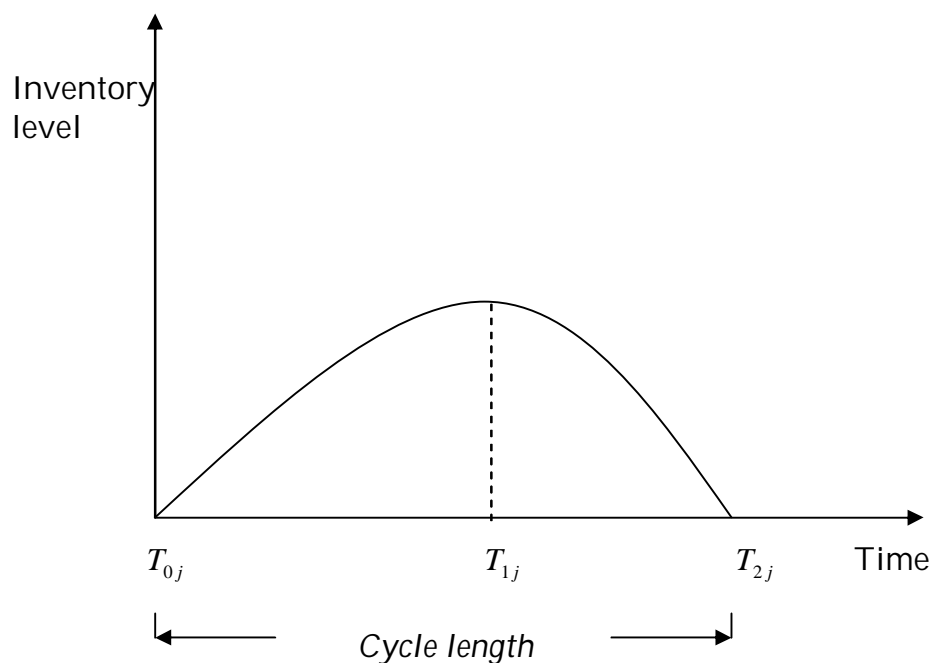


Figure 5-2. Inventory variation of an (EPQ) model under learning-forgetting effects for one cycle

## 5.5 Model formulation

The changes in the inventory level depicted in Figure 5-2, are given by the following differential equations

$$\frac{dI_j(t)}{dt} = P(t) - D(t) - \delta(t)I_j(t) \quad T_{0j} \leq t < T_{1j} \quad (5-14)$$

with the initial condition  $I_j(T_{0j}) = 0$ , and

$$\frac{dI_j(t)}{dt} = -D(t) - \delta(t)I_j(t) \quad T_{1j} \leq t \leq T_{2j} \quad (5-15)$$

with the ending condition  $I_j(T_{2j}) = 0$ .

The solutions of the above differential equations are

$$I_j(t) = e^{-g(t)} \int_{T_{0j}}^t [P(u) - D(u)] e^{g(u)} du \quad T_{0j} \leq t < T_{1j} \quad (5-16)$$

$$I_j(t) = e^{-g(t)} \int_t^{T_{2j}} D(u) e^{g(u)} du \quad T_{1j} \leq t \leq T_{2j} \quad (5-17)$$

respectively, where

$$g(t) = \int \delta(t) dt \quad (5-18)$$

Let  $I(t_1, t_2) = \int_{t_1}^{t_2} I(u) du$ , then from (5-16) and (5-17) we have

$$I_j(T_{0j}, T_{1j}) = \int_{T_{0j}}^{T_{1j}} e^{-g(t)} \left( \int_{T_{0j}}^t [P(u) - D(u)] e^{g(u)} du \right) dt \quad (5-19)$$

$$I_j(T_{1j}, T_{2j}) = \int_{T_{1j}}^{T_{2j}} e^{-g(t)} \left( \int_t^{T_{2j}} D(u) e^{g(u)} du \right) dt \quad (5-20)$$

respectively.

Now using integration by parts, E.q (5-19) is reduced to

$$I_j(T_{0j}, T_{1j}) = \int_{T_{0j}}^{T_{1j}} [G(T_{1j}) - G(u)] [P(u) - D(u)] e^{g(u)} du \quad (5-21)$$

Also, by similar way, E.q (5-20) yields

$$I_j(T_{1j}, T_{2j}) = \int_{T_{1j}}^{T_{2j}} [G(u) - G(T_{1j})] D(u) e^{g(u)} du \quad (5-22)$$

where

$$G(t) = \int e^{-g(t)} dt \quad (5-23)$$

Note that we can set  $T_{0j} = 0$  without loss of generality. The cost components for the given inventory system are as follows:

$$\text{Items cost} = c \int_0^{T_{1j}} P(u) du$$

Note that this cost includes the deterioration cost.

$$\text{Holding cost} = h [ I_j(0, T_{1j}) + I_j(T_{1j}, T_{2j}) ] .$$

Thus, the total cost per unit time of the underlying inventory system during the cycle  $[0, T_{2j}]$ , which consists of the production costs, inventory holding cost, and set up cost, as a function of  $T_{1j}$  &  $T_{2j}$ , say  $Z(T_{1j}, T_{2j})$  is given by

$$Z(T_{1j}, T_{2j}) = \frac{1}{T_{2j}} \left\{ c \int_0^{T_{1j}} P(u) du + h \left[ \int_0^{T_{1j}} \{G(T_{1j}) - G(u)\} \{P(u) - D(u)\} \times e^{g(u)} du + \int_{T_{1j}}^{T_{2j}} \{G(u) - G(T_{1j})\} D(u) e^{g(u)} du \right] + k \right\} \quad (5-24)$$

where  $g(u)$  is given by (5-18), and  $G(u)$  is given by (5-23).

Our goal is to find  $T_{1j}$  &  $T_{2j}$ , that minimize  $Z(T_{1j}, T_{2j})$  where  $Z(T_{1j}, T_{2j})$  is given by (5-24) where the variables  $T_{1j}$  &  $T_{2j}$  are related to each other through the following relations

$$0 < T_{1j} < T_{2j} \quad (5-25)$$

$$e^{-g(T_{1j})} \int_0^{T_{1j}} [P(u) - D(u)] e^{g(u)} du = e^{-g(T_{1j})} \int_{T_{1j}}^{T_{2j}} D(u) e^{g(u)} du \quad (5-26)$$

Relation (5-25) represents the natural monotonicity constraints, since otherwise the given problem, would have no meaning. Relation (5-26) ensures the fact that the inventory levels must have equal values for  $t = T_{1j}$ . Thus, our goal is to solve the following optimization problem, which we shall call problem ( $p$ )

$$(p) = \begin{cases} \text{Minimize } Z(T_{1j}, T_{2j}) & \text{given by (5-24)} & \text{subject to (5-25) and} \\ h_1 = 0 \end{cases}$$

where

$$h_1 = \int_0^{T_{1j}} [P(u) - D(u)] e^{g(u)} du - \int_{T_{1j}}^{T_{2j}} D(u) e^{g(u)} du$$

As mentioned in the previous chapter, the solution of similar problem has been introduced in details in Balkhi [2, 3, and 4]. The most important point to be reminded from Balkhi [2, 3, and 4] is that if we temporarily ignore the monotony constraints (5-25) and call the resulting problem as ( $p_1$ ) then (5-25) do satisfy any solution of ( $p_1$ ). Hence ( $p$ ) and ( $p_1$ ) are equivalent.

Next, we shall introduce the relation between learning and forgetting.

### 5.6 The learn-forget-learn (L-F-L) relationship

Let  $\beta_j$  denotes the amount of equivalent units of experience remembered at the beginning of production run  $j$ , with the initial amount  $\beta_j = 0$ . In each production run  $j$  ( $j=1,2,\dots$ ), the system starts production at time  $T_{0j}$ , in which learning takes place, and hence, the amount  $\beta_j$  will be increased due to the learning effect up to time  $T_{1j}$ , where production ceases, and a maximum level of equivalent units of experience  $Q_j$  is reached, where  $Q_j$  is the number of units produced in cycle  $j$ . At this time the forgetting phenomenon starts its influence, and hence, the amount of equivalent units of experience  $Q_j$  will be decreased due to the forgetting influence up to time  $T_{2j}$ , by which an equivalent amount  $\beta_{j+1}$  is reached, ( $\beta_{j+1} \leq Q_j$ ) and the recommencement for the next production is restarted. The process is repeated. Further, if we denoted by  $t_{1j}$ , the time required to produce the first unit in cycle  $j$ , then, it is to be noted that in the case where the transmission of learning from cycle to cycle does not occur, then  $\beta_j = \beta_{j+1} = 0$ , and  $t_{1j} = t_{1j+1} = t_{11}$ . Moreover, in the case where there is full transmission of learning from cycle to cycle, then  $\beta_{j+1} = Q_j$  and  $t_{1j} \geq t_{1j+1}$ . The equality is to allow the possibility that if  $r > 0 \Rightarrow t_{1j} \rightarrow 0$  as  $Q_j \rightarrow \infty$ . Figure 5-3 depicts the effect of learning and forgetting on the time required to produce a unit.

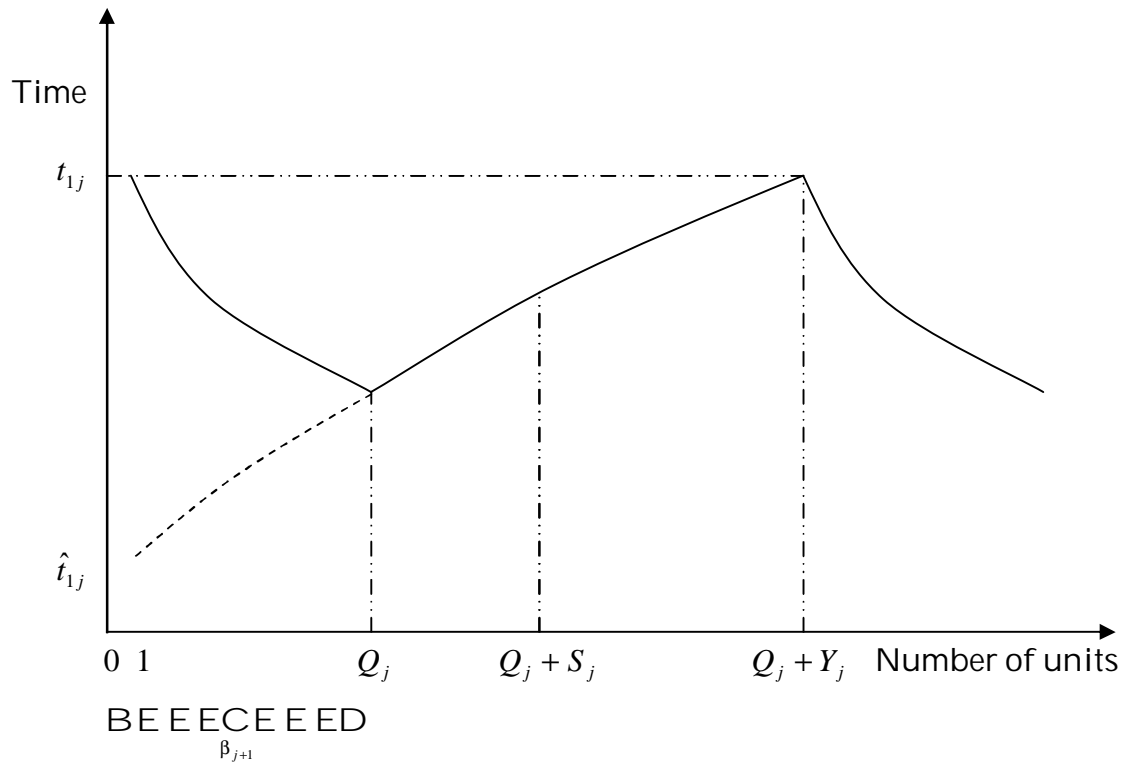


Figure 5-3. The effect of learning and forgetting on the time required to produce a unit

### 5.7 Model formulation under learn –forget –learn (L-F-L)

First, we shall present an approximation form for (5-1). Let  $t_{ij}$  be the time required to produce the  $i^{th}$  unit in the  $j^{th}$  cycle then, from (5-1), we have

$$t_{ij} = t_{1j} i^{-r} \tag{5-27}$$

From (5-27), if  $t_j$  is the time required to produce  $i$  units in the  $j^{th}$  cycle then

$$t_j = \sum_{k=1}^i t_{1j} k^{-r} \approx t_{1j} \int_0^i k^{-r} dk = t_{1j} \frac{i^{1-r}}{1-r} \tag{5-28}$$

Thus if,  $Q_j$  is the amount produced in the interval  $[T_{0j}, T_{1j}]$  then, from (5-28), we have

$$T_{1j} - T_{0j} = t_{1j} \frac{Q_j^{1-r}}{1-r} \quad (5-29)$$

Also if,  $S_j$  is the amount produced in the interval  $[T_{1j}, T_{2j}]$  assuming that there had been no intermittence in production then, from (5-28), we have

$$T_{2j} - T_{0j} = t_{1j} \frac{(Q_j + S_j)^{1-r}}{1-r} \quad (5-30)$$

Next, we present an approximation form for (5-2). Let  $\hat{t}_{\mu j}$  be the equivalent time to produce the  $\mu^{th}$  unit in the  $j^{th}$  cycle then, from (5-2), we have

$$\hat{t}_{\mu j} = \hat{t}_{1j} \mu^{l_j} \quad (5-31)$$

Recalling the definition of  $\beta_{j+1}$ , the time required to produce the first unit in the next production run  $j$  can be found from (5-27) in terms of  $\beta_{j+1}$  as follows:

$$t_{1j+1} = t_{1j} (\beta_{j+1} + 1)^{-r} \quad (5-32)$$

The amount of equivalent units of experience by time  $T_{1j}$ , where production ceases, is equal to the same amount at the beginning of the forgetting phase. Namely,  $Q_j$  the amount produced in the interval  $[T_{0j}, T_{1j}]$ . This can be obtained by equating E.qs (5-27) for  $i = Q_j$  and (5-31) for  $\mu = Q_j$  yielding

$$t_{1j} Q_j^{-r} = \hat{t}_{1j} Q_j^{l_j} \quad (5-33)$$

From which we obtain

$$\hat{t}_{1j} = t_{1j} Q_j^{-(r+l_j)} \quad (5-34)$$

Substituting (5-34) in (5-31) we obtain

$$\hat{t}_{1j} = t_{1j} Q_j^{-(r+l_j)} \mu^{l_j} \quad (5-35)$$

The amount of equivalent units of experience by time  $T_{2j}$ , where the forgetting phase ceases, is equal to the amount  $\beta_{j+1}$ . This can be obtained by equating E.qs (5-31) for  $\mu = (Q_j + S_j)$  and (5-27) for  $i = \beta_{j+1}$  yielding

$$\hat{t}_{1j} (Q_j + S_j)^{l_j} = t_{1j} \beta_{j+1}^{-r} \quad (5-36)$$

From which and (5-34), it follows that

$$t_{1j} \beta_{j+1}^{-r} = t_{1j} Q_j^{-(r+l_j)} (Q_j + S_j)^{l_j}$$

From which we obtain

$$\beta_{j+1} = \left[ \frac{(Q_j + S_j)^{l_j}}{Q_j^{r+l_j}} \right]^{\frac{-1}{r}} \quad (5-37)$$

Let  $\psi(Q_j) = Y_j$  be the corresponding quantity produced during the interval  $[T_{1j}, t_{1j}]$ , where  $t_{1j} \geq T_{2j}$  assuming that there had been no intermittence in production then, from (5-28), we have

$$t_{1j} - T_{0j} = t_{1j} \frac{(Q_j + Y_j)^{1-r}}{1-r} \quad (5-38)$$

Suppose that,  $t_{1j}$  is the time where the equivalent amount of  $Q_j$  units of experience is totally forgotten. The amount of equivalent units of experience by time  $t_{1j}$ , can be obtained by equating E.qs (5-31) for  $\mu = (Q_j + Y_j)$  and (5-27) for  $i = 1$  yielding

$$\hat{t}_{1j} (Q_j + Y_j)^{l_j} = t_{1j} \quad (5-39)$$

Substituting (5-34) in (5-39) we obtain

$$Q_j^{-(r+l_j)} (Q_j + Y_j)^{l_j} = 1$$

Taking the logarithm of both sides, we obtain

$$l_j = \frac{r \log(Q_j)}{\log(Q_j + Y_j) - \log(Q_j)} \quad (5-40)$$

Now, from the above results, and when the system is subject to forgetting, the production rate  $P(t)$ , is equivalently given by

$$P(t) = \frac{\text{Number of units remembered up to time } t}{t} \quad (5-41)$$

For  $T_{0j} = 0$ , (5-29) implies

$$T_{1j} = t_{1j} \frac{Q_j^{1-r}}{1-r} \quad (5-42)$$

Note that the right hand side (RHS) of (5-42) is an increasing function of  $Q_j$ , so  $T_{1j}$  can be uniquely determined as a function of  $Q_j$ , say

$$T_{1j} = f_{1j}(Q_j) \quad (5-43)$$

In addition, from (5-26), we have

$$\int_0^{T_{1j}} [P(u)] e^{g(u)} du = \int_0^{T_{2j}} D(u) e^{g(u)} du \quad (5-44)$$

From which and (5-43) we find that  $T_{2j}$  can be uniquely determined as a function of  $T_{1j}$ , hence of  $Q_j$ , say

$$T_{2j} = f_{2j}(Q_j) \quad (5-45)$$

Thus, if we substitute (5-26), (5-43), and (5-45) in (5-24), then, problem (p) will be converted to the following unconstrained problem with the variable  $Q_j$ , (which we shall call problem (p<sub>2</sub>))

$$\begin{aligned} \text{minimize } W(Q_j) = & \frac{1}{f_{2j}} \left\{ c \int_0^{f_{1j}} P(u) du \right. \\ & \left. + h \left[ \int_0^{f_{1j}} -G(u)\{P(u) - D(u)\}e^{g(u)} du + \int_{f_{1j}}^{f_{2j}} G(u)D(u) e^{g(u)} du \right] + k \right\} \end{aligned} \quad (5-46)$$

The unconstrained case happens since (5-43) and (5-45) were obtained by direct substitution from one in another.

Now, the necessary condition for having a minimum for problem ( $p_2$ ) is

$$\frac{dW}{dQ_j} = 0 \quad (5-47)$$

$$\text{Let } W = \frac{w}{f_{2j}} \text{ then } \frac{dW}{dQ_j} = \frac{w'_{Q_j} f_{2j} - f'_{2j, Q_j} w}{f_{2j}^2}$$

Where  $w'_{Q_j}$  and  $f'_{2j, Q_j}$  are the derivatives of  $w$  and  $f_{2j}$  with respect to (w. r. t)

$Q_j$

Note that (5-47) is equivalent to

$$w'_{Q_j} f_{2j} = f'_{2j, Q_j} w \quad (5-48)$$

Taking the first derivative of both sides of (5-26), (w.r.t)  $Q_j$  we have

$$f'_{1j, Q_j} P(f_{1j}) e^{g(f_{1j})} = f'_{2j, Q_j} D(f_{2j}) e^{g(f_{2j})} \quad (5-49)$$

From which, (5-43), (5-45), and (5-46), we obtain

$$w'_{Q_j} = f'_{1j, Q_j} P(f_{1j}) \left( c + h e^{g(f_{1j})} [G(f_{2j}) - G(f_{1j})] \right) \quad (5-50)$$

From (5-48) through (5-50) we have

$$\frac{f_{2j} f'_{1j, Q_j} P(f_{1j}) \left( c + h e^{g(f_{1j})} [G(f_{2j}) - G(f_{1j})] \right)}{f'_{2j, Q_j}} = w \quad (5-51)$$

$$\text{Also, (5-47)} \Leftrightarrow W = \frac{w}{f_{2j}} = \frac{w'_{Q_j}}{f'_{2j,Q_j}} \quad (5-52)$$

where  $W$  is given by (5-46) and  $w'_{Q_j}$  is given by (5-50). Recalling (5-28), (5-32), and the definition of  $P(t)$  as introduced in (5-41), then we have

$$P(t_{1j}) = \frac{1}{t_{1j}} \quad (5-53)$$

$$P(f_{1j}) = \frac{Q_j}{f_{1j}} \quad (5-54)$$

$$P(f_{2j}) = \frac{\beta_{j+1}}{t_{\beta_{j+1}}} \quad (5-55)$$

where  $t_{\beta_{j+1}}$  is the equivalent time required to produce  $\beta_{j+1}$  units. It is worth noting here, that in the case where the transmission of learning from cycle to cycle does not occur then,  $P(t_{1j}) = \frac{1}{t_{1j}} = P(t_{11}) = \frac{1}{t_{11}}$ . Further, if we denoted by  $f_{1j}$ , the time required to produce  $Q_j$  units in cycle  $j$ , then, this time remains constant in each cycle. Hence,  $P(f_{1j}) = P(f_{11}) = \frac{Q_1}{f_{11}}$ . On the other hand, in the

case where there is full transmission of learning from cycle to cycle, then,

$$P(f_{2j}) = P(f_{1j}) = \frac{Q_j}{f_{1j}}.$$

The two equations (5-51) & (5-52) can, now be used to determine the optimal value of  $Q_j$ . Then, the minimum total cost can be determined from  $W = \frac{w}{f_{2j}}$ .

This will be illustrated in the following example.

## 5.8 Illustrative Example for the (L-F-L) case

The application of the above theoretical results is illustrated in the following example. Consider a production lot-size inventory model with a linear demand rate function given by

$D(t) = at + d$ ,  $d > 0$ ,  $t \geq 0$ . The parameter “ $a$ ” represents the rate of change in the demand rate. The case  $a = 0$  allows the possibility for a constant demand rate, where then  $D(t) = d \forall t, t \geq 0$ . Note also, that  $D(0) = d$  represents the demand rate at time  $t = 0$ . And a deterioration rate given by

$$\delta(t) = \frac{a_1}{b_1 - b_2 t}, \quad t \geq 0, \quad b_1 \geq a_1 \geq 0 \quad \text{and} \quad b_1 > b_2 \geq 0$$

Here  $b_1$  is to be taken sufficiently larger than  $b_2$  in order to keep  $\delta(t) \geq 0$  (viz  $b_1/b_2 \geq T_{2j}$ ). Note that  $\delta(t)$  is an increasing function of  $t$ . The parameters  $a_1, b_1$ , and  $b_2$  are just function parameters so that  $a_1/b_1$  represents the deterioration rate at time  $t = 0$ . If  $b_2 = 0$  then  $\delta(t) = a_1/b_1, \forall t, t \geq 0$ , which means that we have a constant rate of deterioration. If  $a_1 = 0$  then  $\delta(t) = 0, \forall t, t \geq 0$ , which corresponds to the without deterioration case.

Next, we calculate the theoretical functions  $g(t), G(t), Q_j, \int P(t)dt, f_{ij}, f'_{ij, Q_j}$  ( $i=1 \& 2$ ), and  $w$  as they are defined in the previous sections for the above demand and deterioration rates.

Now from (5-18) we have

$$g(t) = \int \delta(t) dt = \int \frac{a_1}{b_1 - b_2 t} dt = \ln(b_1 - b_2 t)^{\frac{-a_1}{b_2}}$$

(All constants of the indefinite integrals are dropped since we are going to use them in the finite case).

$$e^{g(t)} = e^{\ln(b_1 - b_2 t)^{\frac{-a_1}{b_2}}} = (b_1 - b_2 t)^{\frac{-a_1}{b_2}}$$

$$G(t) = \int e^{-g(t)} dt = \int (b_1 - b_2 t)^{\frac{a_1}{b_2}} dt = c_1 (b_1 - b_2 t)^{\frac{a_1 + b_2}{b_2}}, \text{ where } c_1 = \frac{-1}{a_1 + b_2}$$

$$G(t)e^{g(t)} = c_1 (b_1 - b_2 t)$$

Now, from (5-32) and (5-33) we have

$$f_{1j} = t_{1j} \frac{Q_j^{1-r}}{1-r} \tag{5-56}$$

From which, (5-28) , and (5-54)

$$\int P(t_j) dt_j = \int (1-r) di = (1-r) i, \text{ from which, and (5-34), we have}$$

$$\int_0^{f_{1j}} P(u) e^{g(u)} du = \int_0^{f_{2j}} D(u) e^{g(u)} du$$

$$\Rightarrow (1-r) \int_0^{f_{1j}} (b_1 - b_2 u)^{\frac{-a_1}{b_2}} du = \int_0^{f_{2j}} (au + d)(b_1 - b_2 u)^{\frac{-a_1}{b_2}} du$$

To facilitate calculations we can assume (without loss of generality) that

$$-\frac{a_1}{b_2} = \gamma, \text{ where } \gamma \text{ is an integer value. For instance, let us assume that } \gamma = -1,$$

then the last relation leads to

$$(1-r) \ln \left[ \frac{b_1}{b_1 - b_2 f_{1j}} \right] = c_2 \ln \left[ \frac{b_1}{b_1 - b_2 f_{2j}} \right] - a f_{2j}, \text{ where } c_2 = \frac{ab_1 + db_2}{b_2}$$

From which and (5-56) we can find  $f_{2j}$

Now, from the above relations we have

$$f'_{1j,Q_j} = t_{1j} Q_j^{-r}$$

$$f'_{2j,Q_j} = \left[ \frac{(1-r)(b_2 f'_{1j,Q_j})(b_1 - b_2 f_{2j})}{(b_1 - b_2 f_{1j})(c_2 b_2 - a(b_1 - b_2 f_{2j}))} \right]$$

Finally, from (5-46) and with some algebra, using the calculated theoretical functions we find

$$w = c(1-r)f_{1j} + h \left\{ -c_1(1-r) \left( b_1 f_{1j} - \frac{b_2 f_{2j}^2}{2} \right) + c_1 \left( \frac{-ab_2 f_{2j}^3}{3} + \left( \frac{ab_1 - b_2 d}{2} \right) f_{2j}^2 + b_1 d f_{2j} \right) \right\} + k$$

Now, the above results are to be substituted in (5-51) and (5-52) in order to find the solution of the given example.

## 5.9 Numerical verification and Concluding Remarks

The above illustrative example has been verified for a wide range of the model parameters from which we have chosen the following set of values.

The slope of the learning curve	$r = 0.075$
The time required to produce the first unit in the first cycle	$t_{11} = 0.006 \text{ year}$
Unit production cost	$c = \$50$
Unit holding cost per year	$h = \$0.1$
Set-up cost per set-up	$k = \$200$
Parameters of demand rate	$a = 350 \text{ units / year}, d = 160 \text{ units / year}$

Parameters of deterioration rate  $a_1 = b_2 = 10 \text{ units / year}$ ,  $b_1 = 30 \text{ units / year}$

$$\psi(Q_j) = 50Q_j.$$

A Nonlinear programming package has been used to determine the optimal values of  $Q_j, T_{1j}, T_{2j}, l_j, \beta_{j+1}, t_{1j}, P(f_{1j}), P(f_{2j})$  and the corresponding total minimum cost for five successive cycles. The results are shown in table 5-2. In the first cycle, we have taken  $t_{11} = 0.006 \text{ year}$ , which results in a total number of  $Q_1 = 23.16973$  units. Substituting  $\psi(Q_1) = Y_1 = 50Q_1$  in E.q (5-40), we found  $l_1 = 0.05995$ , which corresponds to a 95.93% forgetting rate. From E.q (5-30), we found  $Q_1 + S_1 = 64.689378$  units. Then from E.q (5-37), we found that the amount of units remembered  $\beta_2 = 10.197361$ . Finally from E.q (5-32), we found that the time required to produce the first unit in the second cycle, namely unit number 24.16973, is equal to  $t_{24.16973} = 0.006(11.197361)^{-0.075} = 0.005006$ .

The same procedure is repeated for the other cycles.

The tabulated results indicate that both the time required to produce the first unit and the total time required to produce the optimum production quantity decrease as the number of production runs increases. The decrease in both the optimum quantity produced and the cycle length, as shown in columns 3, 7 respectively, also reflects these reductions. Such decreases are due to the large increase in the production rate for successive cycles. Note that the units remembered and the ratio  $\beta_j/Q_j$  decrease for successive cycles. These decreases are, in fact, due to the large increase of the length of the

consumption period in which forgetting occurs compared to the production period. The increase of the length of the consumption period, hence, the increase in the number of units deteriorated may justify the slight decrease in the total minimum cost in successive cycles (last column in table 5-2). This also may be justified by the increase in both the holding cost and the forgetting rate that starts with  $\cong 95.93\%$ .

Table 5-2: Optimal results under partial transmission of learning for the illustrative example with  $r = 0.075$ , which corresponds to a 94.93% learning rate.

Cycle no.	Time required to produce the first unit	No. of units produced in $[T_{0j}, T_{1j}]$	No. of units remembered in $[T_{0j}, T_{2j}]$	End of production period	Production rate by time	End of consumption Period	Production rate by time	Forgetting slope	Minimum total cost
$j$	$t_{1j}$	$Q_j$	$\beta_{j+1}$	$T_{1j}$	$T_{1j}$	$T_{2j}$	$T_{2j}$	$l_j$	$W$
1	0.006	23.169730	10.197361	0.118730	195.145962	0.306923	183.496150	0.059950	4395.91
2	0.005006	23.014340	8.671330	0.098446	233.776464	0.305357	217.273781	0.059822	4393.20
3	0.004223	22.894910	7.457942	0.082649	277.013602	0.304149	254.663733	0.059723	4391.14
4	0.003598	22.801384	6.477767	0.070151	325.033164	0.303202	295.758671	0.059644	4389.54
5	0.003094	22.726247	5.676414	0.060142	377.879011	0.302445	340.546953	0.059581	4388.19

## 5.10 Viewpoint <sup>a</sup>

In the numerical example for the infinite planning horizon model presented by Jaber and Bonney [15], the incorporation of learning and forgetting appears only in the first cycle. Because when  $t_{1i}$  (the time required to produce the first unit in the next production run) is used to calculate the cumulative time  $t(q_i)$ , to produce  $q_i$  units, it gives different values than those presented by Jaber and Bonney [15].

It is also to be noted that in Jaber and Bonney's [14] model, the shortages occur only in the first cycle, from which we conclude that this model has two phases. The first phase happens in the first cycle in which shortages are allowed, since demand rate is greater than production rate. Then, due to learning effect (and hence there is an increase in the production rate) the backordered demand will be filled starting from the time that the production rate exceeds the demand rate. The second phase does not allow shortages to occur, since production rate is greater than demand rate. This model starting from the second cycle becomes the model presented by Jaber and Bonney [12] for the infinite planning horizon. Therefore, this model does not reflect the incorporation of learning and forgetting effects for the with shortages case.

In fact, we feel that there are some shortcomings in the right meaning of  $\beta_j$  in some previous work. The shortcomings come from the fact that  $\beta_j$  has been treated in previous work ([15], [14]) as the number of units produced instead of units remembered. Besides, in ([12], [15], and [14]) the total forgetting is

assumed to be constant for each production run  $j$ . However, this is not the right case, because  $\beta_j$  may increase as the number of production run increases, from which the forgetting rate decreases for successive production run  $j$ , regardless of the length of non-production periods. The definition of  $\beta_j$  as introduced for our model, coincides with the fact that the minimum break, to which the manufacturer assumes total forgetting is very large compared to the non-production period, in which the maximum loss of experience will not exceed  $Q_j$ , the number of units produced in cycle  $j$ . In fact, we need not consider the cumulative  $\beta_j \leq \sum_{i=1}^{j-1} Q_i$ . The justification of this is that the learning rate is fixed in each production run  $j$ . Hence, we only need to know the time required to produce the first unit in the next production run. Once we found  $\beta_{j+1}$ , hence,  $t_{1j+1}$ , then the total experience gained is included into  $t_{1j+1}$ . This is, indeed, equivalent to saying that a new system has just started with a new time required to produce the first unit, i.e.

$$t_{1j+1} = t_{11}.$$

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<sup>a</sup> , This viewpoint is due to the author.

## 5.11 Conclusion

In this chapter, we first reviewed some new studies related to the effect of learning and forgetting on the optimal production lot-size. Then we have generalized and reformulated the mathematical model for the learning-forgetting- learning (L-F-L) model presented by Jaber and Bonney [12]. The generality of the model comes from the fact that the production, demand and deterioration rates are arbitrary and known functions of time. The minimum break, to which the manufacturer assumes total forgetting has been considered as a function of the optimum quantity produced so that it allows variable total forgetting breaks. We believe that this is reasonable since total forgetting breaks should vary from cycle to cycle, hence, it must be dependent on the optimum quantity produced. A new definition of the production rate is introduced as the number of units remembered per unit time. The definition presented herein, may be applicable to many areas in which the units remembered are treated as the performance level at any given time. An illustrative example for the L-F-L case, which explains the application of the theoretical results under partial transmission of learning and a numerical verification of this illustrative example, is also given. The numerical results clearly reflected the incorporated learning and forgetting effects in the proposed model. This seems to be the first time where such general (L-F-L) model is investigated and numerically verified. A viewpoint that explains the shortcomings in previous studies is also included in this chapter.

The (L-F-L) model derived above can be used as the initial model for further generalizations, which may include:

1. Allow shortages (however, we believe it must be an approximated one).
2. Advanced generalization may be achieved by establishing a new curve that coincides with bounded learning curve presented by De Jong [18], with and without shortages
3. Treating the same (L-F-L) model, but in the finite planning horizon case, with and without shortages.

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